smfprob9(13a).tex

**SMF PROBLEMS 9. 13.12.2013**

Q1. Recall Edgeworth's Theorem (IV.3, D8) that for the multivariate normal distribution $N(\mu, \Sigma)$ with mean vector $\mu$ and covariance matrix $\Sigma$, the density has the form $const. \exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}$. For vectors $y$, $u$, one has

$$y|u \sim N(X\beta + Zu, R), \qquad u \sim N(0, D).$$

Show that

$$u|y \sim N(\mu, \Sigma), \quad \mu = (Z^T R^{-1} Z + D^{-1})^{-1} Z^T R^{-1}(y - X\beta), \quad \Sigma = (Z^T R^{-1} Z + D^{-1})^{-1}.$$

(The apparently complicated algebra should not suggest that this example is artificial! It is standard theory when dealing with *random effects* in regression, and originally arose in statistical studies of breeding of dairy cattle. For background and details, see e.g. [BF], 208-9.)

Q2 (*Inverse of a partitioned matrix*).
Show that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{pmatrix}, \quad M := (A - BD^{-1}C)^{-1}.$$

Q3 (*Conditional independence and the concentration matrix*).
Show that two components $x_i$, $x_j$ of a multinormal vector are conditionally independent given the other components iff $k_{ij} = 0$, where $K = (k_{ij}) = \Sigma^{-1}$ is the concentration matrix. (Take $i = 1$, $j = 2$, and $x_1$ in IV.6 as the sub-vector of the first two components.)

NHB