smfsoln9.tex

SMF SOLUTIONS 7. 31.5.2013

Q1. From the model equation

$$y_i = \sum_{j=1}^p a_{ij}\beta_j + \epsilon_i, \qquad \epsilon_i \quad iid \quad N(0, \sigma^2),$$

the likelihood and log-likelihood are

$$\begin{aligned} L &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}} . \prod_{i=1}^n \exp\{-\frac{1}{2}(y_i - \sum_{j=1}^p a_{ij}\beta_j)^2/\sigma^2\} \\ &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}} . \exp\{-\frac{1}{2}\sum_{i=1}^n (y_i - \sum_{j=1}^p a_{ij}\beta_j)^2/\sigma^2\}, \end{aligned}$$

$$\ell := \log L = const - n\log\sigma - \frac{1}{2}[\sum_{i=1}^n (y_i - \sum_{j=1}^p a_{ij}\beta_j)^2]/\sigma^2. \qquad (*)$$

Maximise w.r.t. β_r in $(*)$ (Fisher, MLE) – equivalently, minimise [...]: $\partial\ell/\partial\beta_r = 0$ (Least Squares):

$$\sum_{i=1}^n a_{ir}(y_i - \sum_{j=1}^p a_{ij}\beta_j) = 0 \qquad (r = 1, \ldots, p):$$

$$\sum_{j=1}^p (\sum_{i=1}^n a_{ir}a_{ij})\beta_j = \sum_{i=1}^n a_{ir}y_i.$$

Write $C = (c_{ij})$ for the $p \times p$ matrix $C := A^T A$, which we note is *symmetric*: $C^T = C$. Then

$$c_{ij} = \sum_{k=1}^n (A^T)_{ik}A_{kj} = \sum_{k=1}^n a_{ki}a_{kj}.$$

So this says

$$\sum_{j=1}^p c_{rj}\beta_j = \sum_{i=1}^n a_{ir}y_i = \sum_{i=1}^n (A^T)_{ri}y_i.$$

In matrix notation, this is

$$(C\beta)_r = (A^T y)_r \qquad (r = 1, \ldots, p): \qquad C\beta = A^T y, \qquad C := A^T A. \ (NE)$$

These are the *normal equations*. As A ($n \times p$, with $p << n$) has full rank, A has rank p, so $C := A^T A$ has rank p, so is non-singular. So the normal equations have solution

$$\hat\beta = C^{-1}A^T y = (A^T A)^{-1}A^T y.$$

Multiplying both sides by A,

$$Py = A(A^T A)^{-1} A^T y = A\hat{\beta}.$$

Q2. (i) The roots $\lambda_1, \ldots, \lambda_p$ of the polynomial $\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p$ should lie inside the unit disk.
(ii) Multiply $(*)$ by X_{t-k} for $k \geq 0$ and take expectations: $E[X_t] = 0$, and

$$\gamma_k = cov(X_t, X_{t-k}) = E[X_t X_{t-k}] = \phi_1 E[X_{t-1}X_{t-k}] + \ldots + \phi_p E[X_{t-p}X_{t-k}] + E[\epsilon_t X_{t-k}].$$

As ϵ_t has mean 0 and is independent of X_{t-k}, this gives

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p}.$$

Divide by γ_0:

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}.$$

(iii) General solution $\rho_k = c_1 \lambda_1^k + \ldots + c_p \lambda_p^k$, c_i constants.

Q3. (i)

$$\gamma_0 = var(X_0) = var(X_t) = E[X_t^2] = E[(\epsilon + \theta\epsilon_{t-1})(\epsilon + \theta\epsilon_{t-1})] = \sigma^2(1 + \theta^2),$$

as $E[\epsilon_t^2] = E[\epsilon_{t-1}^2] = \sigma^2$, $E[\epsilon_t \epsilon_{t-1}] = 0$.
(ii)

$$\gamma_1 = E[X_t X_{t-1}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2})] = \sigma^2\theta,$$
$$\gamma_2 = E[X_t X_{t-2}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-2} + \theta\epsilon_{t-3})] = 0,$$

and similarly $\gamma_k = 0$ for $k \geq 2$.
(iii) $\rho_k = \gamma_k / \gamma_0$. So

$$\rho_0 = 1, \qquad \rho_{\pm 1} = \theta/(1 + \theta^2), \qquad \rho_k = 0 \quad \text{otherwise.}$$

Q4. (i) (X_t) is $ARMA(2, 1)$.
(ii) $X_t - X_{t-1} + \frac{1}{4}X_{t-2} = \epsilon_t + \frac{1}{2}\epsilon_{t-1}$; with B the backward shift,

$$\phi(B)X_t = \theta(B)\epsilon_t,$$

where $\phi(\lambda) = 1 - \lambda + \frac{1}{4}\lambda^2 = (1 - \frac{1}{2}\lambda)^2$, with a repeated root at $\lambda = 2$,
$\theta(\lambda) = 1 + \frac{1}{2}\lambda$, root $\lambda = -2$.
All roots are outside the unit disk in the complex λ-plane, so (X_t) is stationary and invertible.

NHB