smfsoln9(13a).tex

SMF SOLUTIONS 9. 13.12.2013

Q1. The relevant densities are

$$f(y|u) = const. \exp\{-\frac{1}{2}(y-X\beta-Zu)^T R^{-1}(y-X\beta-Zu)\}, \quad f(u) = const. \exp\{-\frac{1}{2}u^T D^{-1} u\}.$$

By Bayes' Theorem, $f(u|y) = f(y|u)f(u)/f(y) = f(u,y)/f(y)$.
But the denominator $f(y)$ is just an 'integration constant', so

$$f(u|y) \propto \exp\{-\frac{1}{2}[(y - X\beta - Zu)^T R^{-1}(y - X\beta - Zu) + u^T D^{-1} u]\}.$$

This has the functional form of a multivariate normal (in u). So it *is* a multivariate normal, and we can find *which* one by picking out the matrix Σ in the $u^T\Sigma^{-1}u$ term, and then the vector μ in the $u^T\Sigma^{-1}\mu$ term. So

$$\Sigma^{-1} = Z^T R^{-1} Z + D^{-1} : \qquad \Sigma = (Z^T R^{-1} Z + D^{-1})^{-1};$$

$$Z^T R^{-1}(y-X\beta) = \Sigma^{-1}\mu : \quad \mu = \Sigma Z^T R^{-1}(y-X\beta) = (Z^T R^{-1} Z + D^{-1})^{-1} Z^T R^{-1}(y-X\beta).$$

Random effects in regression. In this model,

$$y|u \sim N(X\beta + Zu, R), \qquad u \sim N(0, D),$$

the $X\beta$ term is as usual in regression, and covers what are now called the *fixed effects*. The Zu term is new, and covers the *random effects* (we can see that u is random as we are given its distribution).

Such situations are common. As mentioned in the Problems, this arose in studies of breeding and yields for dairy cattle. There, the response variable is milk yield. The fixed effects might involve diet, grazing etc. The random effects are the individual animals.

In finance, the response variables might be output, profit or return, market share etc. The fixed effects might be macroeconomic variables: interest rates, trade figures, employment figures etc. The random effects might be the individual firms.

Other situations (see e.g. [BF], S9.1:
Educational studies. Response variables: exam performance; fixed effects: teaching methods, syllabus etc.; random effects: the individual pupils.

Athletics times. Response variables: race times; fixed effects: age, gender, club status (and, though more difficult to measure, training methods, volume, intensity etc.); random effects: the individual athletes.

Q2.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} AM - BD^{-1}CM & -AMBD^{-1} + BD^{-1} + BD^{-1}CMBD^{-1} \\ CM - CM & -CMBD^{-1} + I + CMBD^{-1} \end{pmatrix}.$$

The $(1,1)$ element is I, from the definition of M. For the $(1,2)$ entry, the first and third terms combine to give $-BD^{-1}$, again by definition of M, so the $(1,2)$ element is 0. The $(2,1)$ element is clearly 0. In the $(2,2)$ element, the first and third terms cancel, so the $(2,2)$ element is I. //

Q3.

$$K_{11} = M := \Sigma_{11} - \Sigma_{12}\Sigma_{22}\Sigma_{21},$$
$$K_{12} = -M\Sigma_{12}\Sigma_{22}, \quad \text{so} \quad K_{11}^{-1}K_{12} = -\Sigma_{12}\Sigma_{22}^{-1}.$$

By the last theorem of IV.6 D9, the covariance matrix of $x_1|x_2$ is the partial covariance matrix $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. With Σ the partitioned matrix in Q3, this is M^{-1}, in the notation of Q3. By Edgeworth's theorem, this identifies the concentration matrix K_{11} of $x_1|x_2$ as $K_{11} = M$:

$$K_{11} = M; \qquad M := (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}.$$

If x_1 is a 2-vector, Σ_{11}, K_{11} are 2×2 matrices. Now (III.1)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}:$$

a non-singular 2×2 matrix A is diagonal iff its inverse A^{-1} is diagonal. For a Gaussian vector, two components are independent iff they are uncorrelated, i.e. iff their (2×2) covariance matrix is diagonal. So: for a 2-vector x_1, with $x^T = (x_1^T, x_2^T)$: the components of x_1 are conditionally independent given x_2 (i.e., given all the other components of x) iff K_{11} is diagonal, i.e. iff $k_{12} = 0$ in an obvious notation. Similarly for k_{ij} for any $i \neq j$. So: components x_i, x_j of a random vector $x \sim N(\mu, \Sigma)$ are conditionally independent given the other components iff $k_{ij} = 0$.

NHB