SMF SOLUTIONS 1. 24.10.2014

Q1.

$$logf(X,\mu) = -\frac{1}{2}(X-\mu)^2/\sigma^2 - \frac{1}{2}\log 2\pi - \log\sigma.$$

$$\partial \log f(X,\mu)/\partial\mu = (X-\mu)/\sigma^2.$$

The information per reading is

$$E[\partial \log f/\partial\mu)^2] = E[(X-\mu)^2/\sigma^4] = \sigma^2/\sigma^4 = 1/\sigma^2.$$

So the information in the whole sample is $I = n/\sigma^2$, so the CR bound is $1/I = \sigma^2/n$. But $\bar{X}$ is unbiased (mean μ), with variance σ^2/n, the CR bound. So $\bar{X}$ is efficient for μ.

Q2. Write $v := \sigma^2$.

$$\log f = const - \frac{1}{2}\log v - \frac{1}{2}(X-\mu)^2/v,$$

$$\partial \log f/\partial v = -\frac{1}{2v} + \frac{(X-\mu)^2}{2v^2} = \frac{1}{2v^2}[(X-\mu)^2 - v].$$

The information per reading is

$$E[(\partial \log f/\partial v)^2] = \frac{1}{4v^4}[E\{(X-\mu)^4\} - 2vE[(X-\mu)^2] + v^2].$$

Now $N(\mu,\sigma^2)$ has MGF $M(t) := E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, so $X - \mu$ has MGF $\exp\{\frac{1}{2}\sigma^2 t^2\} = \exp\{-\frac{1}{2}vt^2\}$,

$$M(t) = 1 + \frac{1}{2}vt^2 + \frac{1}{8}v^2t^4 + \ldots = \sum \mu_k t^k/k!, \qquad \mu_k = E[(X-\mu)^k].$$

$k = 4$: $\mu_4/4! = \mu_4/24 = v^2/8$: $\mu_4 := E[(X-\mu)^4] = 3v^2$, $\mu_2 = varX = \sigma^2 = v$. So the information per reading is

$$\frac{3v^2 - 2v.v + v^2}{4v^4} = \frac{1}{2v^2},$$

and the CR bound is $2v^2/n$. But $\frac{1}{n}\sum_1^n(X_i-\mu)^2$ is unbiased (mean $\frac{1}{n}\sum_1^n \sigma^2 = \sigma^2 = v$), with variance

$$\frac{1}{n^2}.n\ var(X-\mu)^2 = \frac{1}{n}\{E[(X-\mu)^4]-(E(X-\mu)^2])^2\} = \frac{1}{n}(3v^2-v^2) = 2v^2/n,$$

the CR bound. So $\frac{1}{n}\sum_1^n(X_i-\mu)^2$ is an efficient estimator for σ^2.

Q3. We know S_u^2 is unbiased for $\sigma^2 = v$. The CR bound is $2v^2/n$, as in Q2. Now (with S^2 the (biased) sample variance) $nS^2/\sigma^2 \sim \chi^2(n-1)$, which has mean $n-1$ (the number of degrees of freedom, df) and variance $2(n-1)$. So $S_u^2 = (n/(n-1))S^2$ has (mean μ and) variance

$$var\ S_u^2 = \frac{n^2}{(n-1)^2}var\ S^2 = \frac{n^2}{(n-1)^2}.\frac{\sigma^4}{n^2}.2(n-1) = \frac{2\sigma^4}{n-1} = \frac{2v^2}{n-1} = \frac{n}{n-1}.2v^2/n.$$

So the efficiency is $(n-1)/n = 1-1/n \to 1$: S_u^2 is asymptotically efficient for $v = \sigma^2$.

Q4. In an obvious notation,

$$\ell_x = const - \frac{1}{2}n\log|\Sigma| - \frac{1}{2}n\ trace(VS_x) - (\bar{x}-\mu)^T V(\bar{x}-\mu)$$

$$= const - \frac{1}{2}n\log|\Sigma| - \frac{1}{2}n\ trace(VS_x) - \bar{x}^T V\bar{x} - 2\mu^T V\bar{x} + \mu^T V\mu$$

(the two cross-terms are scalars, so are their own transposes, so we can combine them), and similarly for ℓ_y. Subtract:

$$\ell_x - \ell_y = -\frac{1}{2}n\ trace[V(S_x-S_y)] - [\bar{x}^T V\bar{x} - \bar{y}V\bar{y}] - 2\mu^T V(\bar{x}-\bar{y}).$$

This is independent of the parameters μ and Σ (or V) iff

$$\bar{x} = \bar{y}, \qquad S_x = S_y.$$

So by the Lehmann-Scheffé theorem, $(\bar{x}, S_x)$ is minimal sufficient for (μ, Σ).

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