smfsoln10(14).tex

SMF SOLUTIONS 10. 19.12.2014

Q6. (i) Edgeworth's theorem says that if $x \sim N(\mu, \Sigma)$ and $K := \Sigma^{-1}$,

$$f(x) \propto \exp\{-\frac{1}{2}(x-\mu)^T K (x-\mu)\}.$$

$$f(x_1, x_2) \propto \exp\{-\frac{1}{2}(x_1^T - \mu_1^T, x_2^T - \mu_2^T) \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}\},$$

giving (as a scalar is its own transpose, so the two cross-terms are the same)

$$\exp\{-\frac{1}{2}[(x_1^T - \mu_1^T)K_{11}(x_1 - \mu_1) + 2(x_1^T - \mu_1^T)K_{12}(x_2 - \mu_2) + (x_2^T - \mu_2^T)K_{22}(x_2 - \mu_2)]\}.$$

So

$$f_{1|2}(x_1|x_2) = f(x_1, x_2)/f_2(x_2)$$

$$\propto \exp\{-\frac{1}{2}[(x_1^T - \mu_1^T)K_{11}(x_1 - \mu_1) + 2(x_1^T - \mu_1^T)K_{12}(x_2 - \mu_2)\},$$

treating x_2 here as a constant and x_1 as the argument of $f_{1|2}$.

We know (from lectures, though it follows from this proof also) that $x_1|x_2$ is multinormal, so by Edgeworth's theorem again, if the conditional mean of $x_1|x_2$ is ν_1,

$$f_{1|2}(x_1|x_2) \propto \exp\{-\frac{1}{2}(x_1^T - \nu_1^T)V_{11}(x_1 - \nu_1)\},$$

for some matrix V_{11}.

Equating coefficients of the quadratic term gives the conditional concentration matrix of $x_1|x_2$ as $V_{11} = K_{11}$:

$$conc(x_1|x_2) = K_{11}.$$

So the conditional covariance matrix is K_{11}^{-1}.

Then equating linear terms,

$$x_1^T K_{11} \nu_1 = x_1^T K_{11} \mu_1 - x_1^T K_{12}(x_2 - \mu_2): \quad \nu_1 := E[x_1|x_2] = \mu_1 - K_{11}^{-1} K_{12}(x - \mu_2).$$

Problems 9 Q2 for the inverse of a partitioned matrix gives (in an obvious notation)

$$M = K_{11}, \qquad M^{-1} = K_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},$$

$$K_{11}^{-1}K_{12} = M^{-1}(-MBD^{-1}) = -BD^{-1} = -\Sigma_{12}\Sigma_{22}^{-1}.$$

Combining,

$$x_1|x_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$