smfsoln5(14).tex

SMF SOLUTIONS 5. 21.11.2014

Q1. From the model equation

$$y_i = \sum_{j=1}^p a_{ij}\beta_j + \epsilon_i, \quad \epsilon_i \quad iid \quad N(0, \sigma^2),$$

the likelihood and log-likelihood are

$$L = \frac{1}{\sigma^{n} 2\pi^{\frac{1}{2}n}} \cdot \prod_{i=1}^{n} \exp\{-\frac{1}{2}(y_{i} - \sum_{j=1}^{p} a_{ij}\beta_{j})^{2}/\sigma^{2}\}$$

= $\frac{1}{\sigma^{n} 2\pi^{\frac{1}{2}n}} \cdot \exp\{-\frac{1}{2}\sum_{i=1}^{n}(y_{i} - \sum_{j=1}^{p} a_{ij}\beta_{j})^{2}/\sigma^{2}\},$

$$\ell := \log L = const - n\log\sigma - \frac{1}{2} \left[\sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} a_{ij}\beta_j)^2\right] / \sigma^2. \quad (*)$$

Maximise w.r.t. β_r in (*) (Fisher, MLE) – equivalently, minimise [...]: $\partial \ell / \partial \beta_r = 0$ (Least Squares):

$$\sum_{i=1}^{n} a_{ir} (y_i - \sum_{j=1}^{p} a_{ij} \beta_j) = 0 \qquad (r = 1, \dots, p) :$$
$$\sum_{j=1}^{p} (\sum_{i=1}^{n} a_{ir} a_{ij}) \beta_j = \sum_{i=1}^{n} a_{ir} y_i.$$

Write $C = (c_{ij})$ for the $p \times p$ matrix $C := A^T A$, which we note is symmetric: $C^T = C$. Then

$$c_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} a_{ki} a_{kj}.$$

So this says

$$\sum_{j=1}^{p} c_{rj} \beta_j = \sum_{i=1}^{n} a_{ir} y_i = \sum_{i=1}^{n} (A^T)_{ri} y_i$$

In matrix notation, this is

$$(C\beta)_r = (A^T y)_r$$
 $(r = 1, ..., p):$ $C\beta = A^T y,$ $C := A^T A.$ (NE)

These are the normal equations. As A $(n \times p$, with $p \ll n$ has full rank, A has rank p, so $C := A^T A$ has rank p, so is non-singular. So the normal equations have solution

$$\hat{\beta} = C^{-1}A^T y = (A^T A)^{-1}A^T y.$$

Multiplying both sides by A,

$$Py = A(A^T A)^{-1} A^T y = A\hat{\beta}.$$

Q2. (i) The roots $\lambda_1, \ldots, \lambda_p$ of the polynomial $\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1} \lambda - \phi_p$ should lie inside the unit disk. (ii) Multiply (*) by X_{t-k} for $k \ge 0$ and take expectations: $E[X_t] = 0$, and

$$\gamma_k = cov(X_t, X_{t-k}) = E[X_t X_{t-k}] = \phi_1 E[X_{t-1} X_{t-k}] + \ldots + \phi_p E[X_{t-p} X_{t-k}] + E[\epsilon_t X_{t-k}].$$

As ϵ_t has mean 0 and is independent of X_{t-k} , this gives

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p}.$$

Divide by γ_0 :

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}.$$

(iii) General solution $\rho_k = c_1 \lambda_1^k + \ldots + c_p \lambda_p^k$, c_i constants.

Q3. (i)

$$\gamma_{0} = var(X_{0}) = var(X_{t}) = E[X_{t}^{2}] = E[(\epsilon + \theta\epsilon_{t-1})(\epsilon + \theta\epsilon_{t-1})] = \sigma^{2}(1 + \theta^{2}),$$
as $E[\epsilon_{t}^{2}] = E[\epsilon_{t-1}^{2}] = \sigma^{2}, E[\epsilon_{t}\epsilon_{t-1}] = 0.$
(ii)

$$\gamma_{1} = E[X_{t}X_{t-1}] = E[(\epsilon_{t} + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2})] = \sigma^{2}\theta,$$

$$\gamma_{2} = E[X_{t}X_{t-2}] = E[(\epsilon_{t} + \theta\epsilon_{t-1})(\epsilon_{t-2} + \theta\epsilon_{t-3})] = 0,$$

and similarly $\gamma_k = 0$ for $k \ge 2$. (iii) $\rho_k = \gamma_k / \gamma_0$. So

$$\rho_0 = 1, \qquad \rho_{\pm 1} = \theta/(1+\theta^2), \qquad \rho_k = 0 \quad \text{otherwise.}$$

Q4. (i) (X_t) is ARMA(2, 1). (ii) $X_t - X_{t-1} + \frac{1}{4}X_{t-2} = \epsilon_t + \frac{1}{2}\epsilon_{t-1}$; with *B* the backward shift,

$$\phi(B)X_t = \theta(B)\epsilon_t,$$

where $\phi(\lambda) = 1 - \lambda + \frac{1}{4}\lambda^2 = (1 - \frac{1}{2}\lambda)^2$, with a repeated root at $\lambda = 2$, $\theta(\lambda) = 1 + \frac{1}{2}\lambda$, root $\lambda = -2$.

All roots are outside the unit disk in the complex λ -plane, so (X_t) is stationary and invertible.

NHB