SMF SOLUTIONS 7 5.12.2014

Q1 (*Rank-one matrices*). If C is the zero matrix, it has rank 0 – a trivial case, which we exclude.

If C has rank one, the range of C is one-dimensional (this is one of several equivalent definitions of rank). If the domain and range of C have bases e_i, f_j, Ce_i is non-zero for some i (or C would be zero) – w.l.o.g., $Ce_1 = \sum_j c_{1j} f_j \neq 0$. Write $b_j := c_{1j}$: $Ce_1 = \sum_j b_j f_j \neq 0$. As the range of C is one-dimensional, for each i, Ce_i is a multiple $a_i Ce_1$ of Ce_1: $Ce_i = \sum_j a_i b_j f_j$. This says that the linear transformation represented by C has matrix $C = (c_{ij}) = (a_i b_j)$ w.r.t. the bases e_i and f_j.

Conversely, if $C = (a_i b_j)$ is not the zero matrix: at least one $a_i b_j \neq 0$; w.l.o.g., by re-ordering rows and columns, take $a_1, b_1 \neq 0$. Then column j is the multiple b_j/b_1 of column 1, and row i is the multiple a_i/a_1 of row 1. So C has column-rank 1 (only 1 linearly independent column), and row-rank 1, so has rank 1.

Q2.

$$\begin{aligned} AA^{-}A &= ULV^T V L^{-1} U^T U L V^T \\ &= ULL^{-1}LV^T \\ &= ULV^T \\ &= A, \end{aligned}$$

as $U^T U = I$ and $V^T V = I$.

Q3. With $A = (a_1, \ldots, a_n)$ a row of its columns and $x = (x_1, \ldots, x_n)^T$, $Ax = x_1 a_1 + \ldots + x_n a_n$ is a linear combination of the columns of A. So $Ax = b$ can have a solution iff b is a linear combination of the columns of A, i.e. iff adding B to the space spanned by the columns does not increase its dimension, i.e. iff $r((A, b)) = r(A)$.

We know $|A| \neq 0$ is the condition for uniqueness, which is (i). With non-uniqueness but consistency, $Ax_1 = b$ and $Ax_2 = b$ for $x_1 \neq x_2$. Then $A(x_1 - x_2) = 0$, so $A.c(x_1 - x_2) = 0$ for all c, so $A(x_1 + c(x_1 - x_2)) = b$ for all c, giving infinitely many solutions, which is (ii). With inconsistency, there are no solutions, giving (iii).

Q4. (i) With $\mathbf{a}$ the column-vector $(a, 1, 1)^T$, $A = \mathbf{a}\mathbf{a}^T$. So A has rank 1, and as it is 3×3, it has one non-zero eigenvalues and two 0 eigenvalues.

$$A\mathbf{a} = \mathbf{a}\mathbf{a}^T\mathbf{a} = (a^2 + 2)\mathbf{a},$$

as $\mathbf{a}^T\mathbf{a} = a^2 + 2$. This says that A has eigenvalue $a^2 + 2$ with eigenvector $\mathbf{a}$. The other two eigenvalues are 0, with eigenvectors $\mathbf{x}$, $\mathbf{y}$ say. The eigenequation for $\mathbf{x} = (x_1, x_2, x_3)^T$ is

$$ax_1 + x_2 + x_3 = 0,$$

three times (check). Taking $x_3 = 0$, we can take $x_1 = 1$, $x_2 = -a$; taking $x_1 = 0$, we can take $x_2 = 1$, $x_3 = 1$, giving

$$\mathbf{x} = \begin{pmatrix} 1 \\ -a \\ 0 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(ii) $B = A + bI$. So

$$B\mathbf{a} = A\mathbf{a} + bI\mathbf{a} = (a^2 + 2)\mathbf{a} + b\mathbf{a} = (a^2 + b + 2)\mathbf{a}.$$

This says that B has eigenvalue $a^2 + b + 2$ with eigenvector $\mathbf{a}$.

As $A\mathbf{x} = 0$, $B\mathbf{x} = A\mathbf{x} + b\mathbf{x} = b\mathbf{x}$, which says that B has eigenvalue b with eigenvector $\mathbf{x}$. Similarly, B has eigenvalue b with eigenvector $\mathbf{y}$. So:
B has eigenvalues $a^2 + b + 2, b, b$; its eigenvectors are the same as those of A; its rank is 3, unless $b = 0$, when its rank is 1.

NHB