smfsoln1(1617)

SMF SOLUTIONS 1. 26.10.2017

Q1.

$$logf(X,\mu) = -\frac{1}{2}(X-\mu)^2/\sigma^2 - \frac{1}{2}\log 2\pi - \log\sigma.$$

$$\partial \log f(X,\mu)/\partial\mu = (X-\mu)/\sigma^2.$$

The information per reading is

$$E[\partial \log f/\partial\mu)^2] = E[(X-\mu)^2/\sigma^4] = \sigma^2/\sigma^4 = 1/\sigma^2.$$

So the information in the whole sample is $I = n/\sigma^2$, so the CR bound is $1/I = \sigma^2/n$. But $\bar{X}$ is unbiased (mean μ), with variance σ^2/n, the CR bound. So $\bar{X}$ is efficient for μ.

Q2. Write $v := \sigma^2$.

$$\log f = const - \frac{1}{2}\log v - \frac{1}{2}(X-\mu)^2/v,$$

$$\partial \log f/\partial v = -\frac{1}{2v} + \frac{(X-\mu)^2}{2v^2} = \frac{1}{2v^2}[(X-\mu)^2 - v].$$

The information per reading is

$$E[(\partial \log f/\partial v)^2] = \frac{1}{4v^4}[E\{(X-\mu)^4\} - 2vE[(X-\mu)^2] + v^2].$$

Now $N(\mu,\sigma^2)$ has MGF $M(t) := E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, so $X-\mu$ has MGF $\exp\{\frac{1}{2}\sigma^2 t^2\} = \exp\{-\frac{1}{2}vt^2\}$,

$$M(t) = 1 + \frac{1}{2}vt^2 + \frac{1}{8}v^2t^4 + \ldots = \sum \mu_k t^k/k!, \qquad \mu_k = E[(X-\mu)^k].$$

$k=4$: $\mu_4/4! = \mu_4/24 = v^2/8$: $\mu_4 := E[(X-\mu)^4] = 3v^2$, $\mu_2 = varX = \sigma^2 = v$. So the information per reading is

$$\frac{3v^2 - 2v.v + v^2}{4v^4} = \frac{1}{2v^2},$$

and the CR bound is $2v^2/n$. But $\frac{1}{n}\sum_1^n (X_i - \mu)^2$ is unbiased (mean $\frac{1}{n}\sum_1^n \sigma^2 = \sigma^2 = v$), with variance

$$\frac{1}{n^2}.n\ var(X-\mu)^2 = \frac{1}{n}\{E[(X-\mu)^4] - (E(X-\mu)^2])^2\} = \frac{1}{n}(3v^2 - v^2) = 2v^2/n,$$

the CR bound. So $\frac{1}{n}\sum_1^n (X_i - \mu)^2$ is an efficient estimator for σ^2.

Q3. We know S_u^2 is unbiased for $\sigma^2 = v$. The CR bound is $2v^2/n$, as in Q2. Now (with S^2 the (biased) sample variance) $nS^2/\sigma^2 \sim \chi^2(n-1)$, which has mean $n-1$ (the number of degrees of freedom, df) and variance $2(n-1)$. So $S_u^2 = (n/(n-1))S^2$ has (mean μ and) variance

$$var\ S_u^2 = \frac{n^2}{(n-1)^2} var\ S^2 = \frac{n^2}{(n-1)^2}.\frac{\sigma^4}{n^2}.2(n-1) = \frac{2\sigma^4}{n-1} = \frac{2v^2}{n-1} = \frac{n}{n-1}.2v^2/n.$$

So the efficiency is $(n-1)/n = 1 - 1/n \to 1$: S_u^2 is asymptotically efficient for $v = \sigma^2$.

Q4. (i) Consider

$$Q(\lambda) := \int (\lambda f + g)^2 = \lambda^2 \int f^2 + 2\lambda \int fg + \int g^2.$$

This is always non-negative, so its discriminant is ≤ 0 (if it were > 0, there would be two distinct real roots with a sign change between them). So ("$b^2 - 4ac \leq 0$")

$$(\int fg)^2 \leq \int f^2 \int g^2.$$

Equality holds iff Q has a double root, λ_0 say. Then

$$Q(\lambda_0) = \int (\lambda_0 f + g)^2 = 0.$$

This forces

$$\lambda_0 f + g = 0$$

(a.e.), the required linear relation.

(ii) We apply this to $X - E[X]$, $Y - E[Y]$, we find equality in CR iff a linear relation of the form $a(X - E[X]) + b(Y - E[Y]) = 0$ holds. Taking $X = \ell'$ (recall $E[\ell'] = 0$), $Y = u$ (recall $EY = \theta$ as u is unbiased for θ), we find

$$u - \theta = a\ell'$$

for some a (so $b = \theta$).
(iii) So

$$\ell' = u/a(\theta) - \theta/a(\theta) : \qquad \ell = u\int d\theta/a(\theta) - \int \theta d\theta/a(\theta) + k(\mathbf{X}) :$$

$$L(\theta;\mathbf{X}) = \exp\{\alpha(\theta)u(\mathbf{X}) + \beta(\theta) + k(\mathbf{X})\}.$$

Q5.

$$\ell = -n\log 2 - \sum |x_i - \theta|.$$

To maximise this – i.e. minimise $\sum |x_i - \theta|$ – draw a graph. From this, the sum is minimised by $\theta = Med$, and increases linearly on either side of the sample median. So the MLE is $\hat{\mu} = Med$.
(ii) With one reading, as above, ℓ decreases with slope -1 to the right of Med, slope +1 to the left of Med. So $(\ell')^2 = 1$ (except at $\lambda = Med$, where the derivative is not defined – but we are going to integrate, and so can neglect null sets, let alone single points, so this does not matter). So $I = \int (\partial \log f/\partial\theta)^2 f = \int f = 1$, as f is a density. So the CR bound is $1/n$.

We are given that Med is asymptotically normal, and that its mean is $med = \theta$, so Med is asymptotically unbiased. By symmetry, the population median is $med = \theta$, where the density is $\frac{1}{2}$. So $4f(med)^2 = 1$, and the asymptotic variance of the sample median is $1/n$, the CR bound, so Med is also asymptotically efficient.

Q6. (i)

$$f(x;\mu) = \frac{1}{\pi(1 + (x-\mu)^2)}, \qquad \ell = \log f = c - \log[1 + (x-\mu)^2],$$

$$\ell' = \frac{2(x-\mu)}{1 + (x-\mu)^2}, \qquad \ell'(\mathbf{x};\theta) = 2\sum_1^n \frac{(x_i - \mu)}{1 + (x_i - \mu)^2}.$$

But (Q1) we have efficiency iff ℓ' factorises in the form $\ell'(\mathbf{x};\theta) = A(\theta)(u(\mathbf{x}) - \theta)$. The likelihood here does not factorise, so there is no efficient estimator.
(ii) The information per reading is

$$E[(\ell')^2] = \int (\partial f/\partial\mu)^2 f = \frac{4}{\pi}\int \frac{(x-\mu)^2}{[1+(x-\mu)^2]^3}dx = \frac{4}{\pi}\int \frac{x^2}{[1+x^2]^3}dx = \frac{4}{\pi}I,$$

say. One can evaluate I by Complex Analysis ($f(z) := z^2/[1+z^2]^3$, round the contour Γ – semicircle in the upper half-plane on base $[-R, R]$; f has a triple pole inside Γ of residue $-i/16$, so $I = 2\pi i\ Res = \pi/8$), giving the information per reading as $\frac{1}{2}$. So the information in a sample of size n is $n/2$, and the MLE has asymptotic variance $2/n$. As in Q2, the sample median has asymptotic variance $\pi^2/4n$. So the asymptotic efficiency is their ratio, $8/\pi^2 \sim 81\%$. NHB