SMF EXAMINATION 2014-15

Three hours, six questions; do five; 20 marks each

Q1. (i) For the symmetric exponential location family

$$f(x) = \frac{1}{2}\exp\{|x-\theta|\} :$$

(a) Show that

$$\ell = const - \sum |x_i - \theta|.$$

Show that this is maximised where θ is the *median* of the sample, $Med = Med(x_1, \ldots, x_n)$, and deduce that this is the MLE:

$$\hat{\mu} = Med.$$

(b) Show that the information per reading is 1 (use $I = \int (\partial \log f/\partial\theta)^2 f$).

We quote that the sample median Med is asymptotically normal with mean the (population) median med and variance $1/(4nf(med)^2)$.

(c) Show that Med is asymptotically normal, unbiased and efficient.

(ii) For the Cauchy location family

$$f(x;\mu) = \frac{1}{\pi(1+(x-\mu)^2)} :$$

(a) Show that this does not belong to the exponential family.

(b) Show that the MLE has asymptotic variance

$$var(\hat{\mu}) \sim 2/n$$

and efficiency $8/\pi^2$ ($\sim 81\%$). You may quote that

$$I := \int_{-\infty}^{\infty} \frac{x^2}{[1+x^2]^3} dx = \frac{1}{2}.$$

Q2. The heights of two towers A and B are measured, and so is the difference of their heights (B is the higher). The true values are a and b; the readings are y_A, y_B, y_{B-A}. Formulate a regression model.

Find

(a) the design matrix A,

(b) $C := A^T A$, $C^{-1}A^T$,
(c) the projection matrix $P := AC^{-1}A^T$ and $I - P$,
(d) the parameter estimates,
(e) the fitted values,
(f) the ranks of P and $I - P$.
(g) How would you find an unbiased estimator of the error variance σ^2?

Q3. A 3-vector x has the multivariate normal distribution $N(\mu, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.$$

A 2-vector y is defined by $y_1 := x_1 + x_2$, $y_2 := x_2 + x_3$.
(i) Find the distribution of y. (ii) Find the conditional distribution of y_1 given y_2. (You may quote without proof the formula
$x_2|x_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$.)

Q4. In an $ARMA(1,1)$ model

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1},$$

(i) state (without proof) the conditions for stationarity and invertibility.
(ii) Assuming these, find
(a) the variance $\gamma_0 = var\ X_0$;
(b) the covariance $\gamma_k = cov(X_t, X_{t-k})$.

Q5. (i) Describe briefly the main contributions of Markowitz's work to mathematical finance.
(ii) Describe briefly the elliptically contoured model, and specify its parametric part and its non-parametric part.
(iii) What are the principal deficiencies of normal (Gaussian) models in mathematical finance?
(iv) How does the asset return distribution depend on the return period?

Q6. State without proof Edgeworth's theorem for the density of the multinormal law $N(\mu, \Sigma)$, in terms of the concentration matrix $K := \Sigma^{-1}$.

If a multinormal vector x is partitioned into x_1 and x_2, with μ, Σ, K partitioned accordingly, derive the conditional distribution of x_1 given x_2 in terms of μ, K.

Give the same result in terms of μ and Σ.
(You may assume the formula for the inverse of a partitioned matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{pmatrix}, \quad M := (A - BD^{-1}C)^{-1},$$

when all inverses exist.)

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