smfmockexam.tex

SMF MOCK EXAMINATION. 18.2.2011

Q1. (i) State without proof the spectral decomposition for a real symmetric matrix A. **[4]**
(ii) Show how to define the square root and inverse square root of A. **[2, 2]**
(iii) If x has independent $N(0, \sigma^2)$ components and $y := Ox$ with O an orthogonal matrix, show that y has the same distribution as x. **[5]**
(iv) If A is real and symmetric, and $Q := x^T Ax$ is the quadratic form in x as in (iii), express Q as a quadratic form in independent normal variables with diagonal matrix. **[4]**
(v) For A real symmetric, show that A is idempotent iff all its eigenvalues are 0 or 1. **[4]**
(vi) For P a symmetric projection, show that the rank and trace of P coincide. **[4]**

Q2. (i) Give the definition of the multivariate normal distribution $N(\mu, \Sigma)$. **[2]**
(ii) Show that if $x \sim N(\mu, \Sigma)$ and $y := Ax + b$, then y is multivariate normal, and find its mean vector and covariance matrix. **[2, 2, 4]**
(iii) Show that any subvector of a multivariate normal vector is multivariate normal. **[2]**
(iv) If $x \sim N(\mu, \Sigma)$ is partitioned as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

state without proof the conditional distribution of x_1 given x_2. **[3]**
(v) If

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{pmatrix}$$

and $x \sim N(\mu, \Sigma)$, find the conditional distribution of

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | x_3. \qquad \textbf{[10]}$$

Q3. (i) If $X = (X_t)$ is L_1-bounded, i.e. $\|X\|_1 := \sup_t E[|X_t|] < \infty$, and $\psi = (\psi_j) \in \ell_1$ (i.e. $\|\psi\|_1 = \sum_j |\psi_j| < \infty$), show that $\sum_j \psi_j X_{t-j}$ converges a.s. and in ℓ_1. **[5, 5]**

(ii) Show that $\ell_1 \subset \ell_2$. **[5]**

(iii) If also X is L_2-bounded, i.e. $\|X\|_2 := \sup_t E[|X_t|^2] < \infty$, show that $\sum_j \psi_j X_{t-j}$ also converges in ℓ_2, to the same sum. **[10]**

Q4. Describe briefly, without proofs, the method of principal components analysis. **[6]**

Discuss the advantages and disadvantages of working with covariances and with correlations. **[6]**

Give examples, in the financial area, where each might be appropriate. **[13]**

Q5. (i) Show that

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix},$$

where the parameter ρ is a correlation, has eigenvalues $1 + 2\rho$ (simple) and $(1 - 2\rho)$ (double), with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \qquad [4]$$

(ii) Deduce that this matrix can only be a correlation matrix under a restriction on ρ. Find this restriction, and the further restriction that Σ be non-singular. **[4, 4]**

(iii) If $x_i \sim N(\mu, \Sigma)$ and the vector y has coordinates $y_1 := x_1 + x_2$, $y_2 := x_2 + x_3$, find the mean vector and covariance matrix of y. **[4, 9]**

N. H. Bingham