smfmockexam12.tex

SMF MOCK EXAMINATION 2012

Six questions; do five; 20 marks per question

Q1. In a parametric model, write $L(\theta)$ for the likelihood function.
(i) Define the *score function* $s(\theta)$, and the *information* $I(\theta)$. **[2, 2]**
(ii) Show that $s(\theta)$ has mean 0 and variance $I(\theta)$ (assuming any regularity conditions that you need, which should be clearly stated). **[8, 8]**

Q2. In your choice of a Bayesian or a non-Bayesian setting,
(i) define *sufficiency* of a statistics T for a parameter θ; **[4]**
(ii) state and prove the factorisation criterion for a statistic T to be sufficient for θ. **[4, 12]**

Q3. Define the *likelihood ratio test* for H_0 v. H_1, with both hypotheses possibly composite. **[2]**

For a normal family $N(\mu, \sigma^2)$, H_0 is $\mu = \mu_0$, while H_1 is μ unrestricted (the nuisance parameter σ is unknown). Derive the likelihood ratio test, and show that it reduces to a t-test. **[8, 10]**

Q4. In the $AR(2)$ model

$$X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + \epsilon_t,$$

(i) find the moving-average representation. **[14]**
(ii) Write down the Yule-Walker equations, and describe briefly how you would solve them. **[3, 3]**

Q5. (i) If $X = (X_t)$ is L_1-bounded, i.e. $\|X\|_1 := \sup_t E[|X_t|] < \infty$, and $\psi = (\psi_j) \in \ell_1$ (i.e. $\|\psi\|_1 = \sum_j |\psi_j| < \infty$), show that $\sum_j \psi_j X_{t-j}$ converges a.s. and in ℓ_1. **[4, 4]**
(ii) Show that $\ell_1 \subset \ell_2$. **[4]**
(iii) If also X is L_2-bounded, i.e. $\|X\|_2 := \sup_t E[|X_t|^2] < \infty$, show that $\sum_j \psi_j X_{t-j}$ also converges in ℓ_2, to the same sum. **[8]**

Q6. (i) Show that a matrix $A = (a_{ij})$ has rank 1 iff $A = ab^T$, for column vectors A, b, that is, iff $a_{ij} = b_i c_j$ for some b_i, c_j (then A is the *tensor product* of the vectors a and b, $a \otimes b$). **[10]**

(ii) Find the eigenvalues, eigenvectors and rank of the matrix

$$A = \begin{pmatrix} a^2 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{pmatrix}. \qquad [\mathbf{4}, \mathbf{4}, \mathbf{2}]$$

NHB