

## SMF PROBLEMS 2. 19.10.2017

Q1 (*Conditions for equality in the Cramér-Rao (Information, CR) Inequality*).

(i) Show that in the Cauchy-Schwarz Inequality

$$(\int fg)^2 \leq (\int f^2)(\int g^2),$$

equality holds iff there is a linear relationship between  $f$  and  $g$ :

$$af + bg = 0$$

for some constants  $a, b$ .

(ii) Deduce that we have equality in CR iff

$$u = a\ell' + b$$

for some  $a, b$ . Find  $b$ .

(iii) Observing that the constant  $a$  above may depend on the parameter  $\theta$ , and that when we integrate  $\ell'$  to get  $\ell$ ,  $L$  the constant of integration may depend on the data  $\mathbf{X}$ , show that equality holds iff  $L$  has the form

$$L = \exp\{\alpha(\theta)u(\mathbf{X}) + \beta(\theta) + k(\mathbf{X})\}.$$

Such likelihoods form the *exponential family* – roughly, the families for which one can do parameter estimation satisfactorily.

(iv) Show that  $u(\mathbf{X})$  is (a) sufficient for  $\theta$ ; (b) minimal sufficient for  $\theta$ .

Q2 (*Symmetric exponential location family*). Here

$$f(x) = \frac{1}{2} \exp\{|x - \theta|\}.$$

(i) Show that

$$\ell = \text{const} - \sum |x_i - \theta|.$$

Show that this is maximised where  $\theta$  is the *median* of the sample,  $Med = Med(x_1, \dots, x_n)$ , and deduce that this is the MLE:

$$\hat{\mu} = Med.$$

(ii) Show that the information per reading is 1 (use  $I = \int (\partial \log f / \partial \theta)^2 f$ ).

We quote that the sample median  $Med$  is asymptotically normal with mean the (population) median  $med$  and variance  $1/(4nf(med)^2)$ .

(iii) Show that  $Med$  is asymptotically normal, unbiased and efficient.

Q3 (*Cauchy location family*). The Cauchy location family is defined by

$$f(x; \mu) = \frac{1}{\pi(1 + (x - \mu)^2)}.$$

(i) Show that this does not belong to the exponential family (it is a standard example of this!)

(ii) Show that the MLE has asymptotic variance

$$\text{var}(\hat{\mu}) \sim 2/n$$

and efficiency  $8/\pi^2$  ( $\sim 81\%$ ). You may quote that

$$I := \int_{-\infty}^{\infty} \frac{x^2}{[1 + x^2]^3} dx = \frac{1}{2}.$$

NHB