

**SMF PROBLEMS 7. 24.11.2017**

Q1. (i) For a Bernoulli distribution  $B(p)$  with uniform prior on  $p \in [0, 1]$ , show that the posterior distribution is a Beta distribution, and find the parameters.

(ii) Repeat with a Beta prior  $B(\alpha, \beta)$ .

Q2. Find the mean of  $B(\alpha, \beta)$ .

Q3. Hence find the posterior mean in Q1(ii), and interpret this as the sample size  $n$  increases.

Q4 (*Convolutions of Gammas and Euler's integral for the Beta function*). Write  $f_\alpha$  for the exponential density with parameter  $\alpha$ :

$$f_\alpha(x) = x^{\alpha-1}e^{-x}/\Gamma(\alpha) \quad (x > 0).$$

(i) Show that

$$f_\alpha * f_\beta = f_{\alpha+\beta}.$$

(ii) Deduce Euler's integral for the Beta function:

$$B(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Q5. For the shifted exponential distribution with parameter  $\theta > 0$  (density  $e^{-(x-\theta)}$  for  $x > \theta$ ),

(i) find the MLE; (ii) find for each  $n$  a sufficient statistic;

(iii) if  $\theta$  has prior density  $\lambda e^{-\lambda\theta}$  ( $\theta \sim E(\lambda)$ ), find the posterior density to within a multiplicative constant.

Q6. In a mixed-model regression,

$$y|u \sim N(A\beta + Bu, R), \quad u \sim N(0, D).$$

Show that  $u|y$  is also Gaussian,

$$u|y \sim N(\Sigma, \mu),$$

and find  $\Sigma$  and  $\mu$ .

NHB