

smfsoln1(1718)

SMF SOLUTIONS 1. 26.10.2017

Q1.

$$\log f(X, \mu) = -\frac{1}{2}(X - \mu)^2/\sigma^2 - \frac{1}{2} \log 2\pi - \log \sigma.$$

$$\partial \log f(X, \mu)/\partial \mu = (X - \mu)/\sigma^2.$$

The information per reading is

$$E[\partial \log f/\partial \mu]^2] = E[(X - \mu)^2/\sigma^4] = \sigma^2/\sigma^4 = 1/\sigma^2.$$

So the information in the whole sample is $I = n/\sigma^2$, so the CR bound is $1/I = \sigma^2/n$. But $\bar{X}$ is unbiased (mean μ), with variance σ^2/n , the CR bound. So $\bar{X}$ is efficient for μ .

Q2. Write $v := \sigma^2$.

$$\log f = \text{const} - \frac{1}{2} \log v - \frac{1}{2}(X - \mu)^2/v,$$

$$\partial \log f/\partial v = -\frac{1}{2v} + \frac{(X - \mu)^2}{2v^2} = \frac{1}{2v^2}[(X - \mu)^2 - v].$$

The information per reading is

$$E[(\partial \log f/\partial v)^2] = \frac{1}{4v^4}[E\{(X - \mu)^4\} - 2vE[(X - \mu)^2] + v^2].$$

Now $N(\mu, \sigma^2)$ has MGF $M(t) := E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, so $X - \mu$ has MGF $\exp\{\frac{1}{2}\sigma^2 t^2\} = \exp\{-\frac{1}{2}vt^2\}$,

$$M(t) = 1 + \frac{1}{2}vt^2 + \frac{1}{8}v^2t^4 + \dots = \sum \mu_k t^k/k!, \quad \mu_k = E[(X - \mu)^k].$$

$k = 4$: $\mu_4/4! = \mu_4/24 = v^2/8$: $\mu_4 := E[(X - \mu)^4] = 3v^2$, $\mu_2 = \text{var} X = \sigma^2 = v$.
So the information per reading is

$$\frac{3v^2 - 2v \cdot v + v^2}{4v^4} = \frac{1}{2v^2},$$

and the CR bound is $2v^2/n$. But $\frac{1}{n} \sum_1^n (X_i - \mu)^2$ is unbiased (mean $\frac{1}{n} \sum_1^n \sigma^2 = \sigma^2 = v$), with variance

$$\frac{1}{n^2} \cdot n \operatorname{var}(X - \mu)^2 = \frac{1}{n} \{E[(X - \mu)^4] - (E(X - \mu)^2)^2\} = \frac{1}{n} (3v^2 - v^2) = 2v^2/n,$$

the CR bound. So $\frac{1}{n} \sum_1^n (X_i - \mu)^2$ is an efficient estimator for σ^2 .

Q3. We know S_u^2 is unbiased for $\sigma^2 = v$. The CR bound is $2v^2/n$, as in Q2. Now (with S^2 the (biased) sample variance) $nS^2/\sigma^2 \sim \chi^2(n-1)$, which has mean $n-1$ (the number of degrees of freedom, df) and variance $2(n-1)$. So $S_u^2 = (n/(n-1))S^2$ has (mean μ and) variance

$$\operatorname{var} S_u^2 = \frac{n^2}{(n-1)^2} \operatorname{var} S^2 = \frac{n^2}{(n-1)^2} \cdot \frac{\sigma^4}{n^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1} = \frac{2v^2}{n-1} = \frac{n}{n-1} \cdot 2v^2/n.$$

So the efficiency is $(n-1)/n = 1 - 1/n \rightarrow 1$: S_u^2 is asymptotically efficient for $v = \sigma^2$.

NHb