ma414l6.tex
Lecture 6. 16.2.2012

Corollary (Doob). A non-negative supermg X_n is a.s. convergent.

Proof. As X_n is a supermg, EX_n decreases. As $X \geq 0$, $E[X_n] \geq 0$. So $E[|X_n|] = E[X_n]$ is decreasing and bounded below, so (convergent and) bounded: X_n is L_1-bounded. So the submg $-X_n$ is L_1-bounded, so a.s. convergent by Doob's Theorem, so X_n is a.s. convergent.

6. Uniform Integrability (UI) and Martingales (Mgs)

Random variables X_n are called *uniformly integrable* (UI) if

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \downarrow 0 \qquad (a \uparrow \infty).$$

Note that:
(i) If (X_n) are UI, then each X_n is integrable. For,

$$E|X_n| = \int_{\{|X_n|\leq a\}} |X_n| dP + \int_{\{|X_n|>a\}} |X_n| dP \leq a + o(1) < \infty.$$

(ii) If each $|X_n| \leq Y \in L_1$, then (X_n) is UI.
(iii) If $\sup_n |X_n| \in L_1$, then (X_n) is UI, as then

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \leq \int_{\{|X_n|\geq a\}} (\sup_k |X_k|) dP \to 0 \qquad (a \to \infty),$$

by dominated convergence.

The next result extends Fatou's Lemma and dominated convergence.

Theorem. For (X_n) UI and non-negative (or bounded below by an integrable function),
(i) $E[\liminf X_n] \leq \liminf E[X_n] \leq \limsup E[X_n] \leq E[\limsup X_n]$.
(ii) If $X_n \to X$ a.s. or in probability, then $X \in L_1$ and $E[X_n] \to E[X]$.

Proof. (i) As $\limsup f_n$ is the limit of a subsequence of (f_n), integrability of $\limsup f_n$ follows by Fatou's Lemma (I.1, L1). For $c \geq 0$,

$$E[X_n] = \int X_n dP = \int_{\{X_n<-c\}} X_n dP + \int_{\{X_n\geq -c\}} X_n dP.$$

Choose $\epsilon > 0$. By UI, we can take c so large that each first term on RHS has modulus $< \epsilon$. As $X_n I(X_n \geq -c) \geq -c$, integrable, Fatou's Lemma gives

$$\liminf \int_{\{X_n \geq -c\}} X_n dP \geq \int \liminf X_n I(X_n \geq -c) dP.$$

As $X_n I(X_n \geq -c) \geq X_n$, RHS $\geq \int \liminf X_n dP$. Combining,

$$\liminf E[X_n] \geq E[\liminf X_n] - \epsilon.$$

As $\epsilon > 0$ is arbitrarily small, this gives the 'liminf' part; the 'limsup' part is similar.
(ii) If $X_n \to X$ a.s., (ii) follows from (i). If $X_n \to X$ in probability, there is a subsequence $X_{n_k} \to X$ a.s. (quote). Then by (i), $X \in L_1$, and $E[X_{n_k}] \to E[X]$. Similarly, every subsequence has a further sub-subsequence $\to X$ a.s., along which the mean converges to $E[X]$. But this implies convergence along the whole sequence (check). //

Uniform integrability is what is needed to pass from a.s. convergence to L_1-convergence, and to strengthen convergence in prob. to a.s. convergence:

Proposition 1. (i) If X_n is UI and a.s. convergent, it is L_1-convergent.
(ii) If $p \in (0, \infty)$, $X_n \to X$ in probability and $(|X_n|^p)$ is UI, then $X_n \to X$ in L_p.

Proof. (i) For $a > 0$, define $f_a(x)$ as $-a$ for $x \leq -a$, x for $-a \leq x \leq a$, $+a$ for $x \geq a$. Then f_a is bounded and continuous, and (check) $|x - f_a(x)| \leq x$. By the Triangle Inequality,

$$\|X_m - X_n\|_1 \leq \|f_a(X_m) - f_a(X_n)\|_1 + \|X_m - f_a(X_m)\|_1 + \|X_n - f_a(X_n)\|_1.$$

If $X_n \to X_\infty$ a.s., then also $f_a(X_n) \to f_a(X_\infty)$ a.s. as f_a is continuous. As $|f_a| \leq a$, dominated convergence then shows that $f_a(X_n) \to f_a(X_\infty)$ in L_1 (so is Cauchy in L_1). Also

$$\|X_m - f_a(X_m)\|_1 \leq \int_{\{|X_m| > a\}} |X_m| dP$$

by definition of f_a. Let $m, n \to \infty$: the first term on the RHS $\to 0$ as $f_a(X_n)$ is Cauchy in L_1. By UI, the second and third terms $\to 0$ as $a \to \infty$. This

shows that X_n is Cauchy in L_1, so convergent in L_1 as L_1 is complete (Riesz-Fischer theorem – quote). //
(ii) We quote this, as we shall not need it; see e.g. Ash [A], Th. 7.5.4.

Proposition 2. (X_n) is UI iff $E[|X_n|]$ is bounded and (X_n) is uniformly absolutely continuous, i.e.

$$\sup_n \int_A |X_n| dP \to 0 \qquad (P(A) \to 0).$$

Proof. If (X_n) is UI,

$$\int_A |X_n| dP = \int_{A \cap \{|X_n| \geq c\}} |X_n| dP + \int_{A \cap \{|X_n| < c\}} |X_n| dP \leq \int_{\{|X_n| \geq c\}} |X_n| dP + cP(A).$$

Choose $\epsilon > 0$. For c large enough, the first term $< \epsilon/2$ for all n. Then if $P(A) < \epsilon/(2c)$, $\int_A |X_n| dP < \epsilon$, proving (X_n) unif. abs. continuous. Also

$$E|X_n| = \int_{\{|X_n| \geq c\}} |X_n| dP + \int_{\{|X_n| < c\}} |X_n| dP < \epsilon + c$$

for large n (the first term by UI), so $E|X_n|$ is bounded.

Conversely, by Markov's Inequality

$$P(|X_n| \geq c) \leq c^{-1} E|X_n| \leq c^{-1} \sup_n E|X_n| \to 0 \qquad (c \to \infty),$$

uniformly in n. This and the uniform absolute continuity give

$$\int_{\{|X_n| \geq c\}} |X_n| dP \to 0 \qquad (c \to \infty)$$

uniformly in n, giving (X_n) UI. //

Lemma (UI Lemma). If $X \in L_1$, then the family $\{E[X|\mathcal{B}]\}$ as $\mathcal{B}$ varies over all sub-σ-fields of $\mathcal{A}$ is UI.

Proof. $|E[X|\mathcal{B}]| \leq E[|X| \, |\mathcal{B}]$. Also, for $a > 0$ $\{|E[X|\mathcal{B}] \, | > a\} \subset \{E[|X| \, |\mathcal{B}] > a\}$, so $I(\{|E[X|\mathcal{B}] \, | > a\}) \leq I(\{E[|X| \, |\mathcal{B}] > a\})$. Multiply:

$$|E[X|\mathcal{B}] \, | I(\{|E[X|\mathcal{B}] \, | > a\}) \leq E[|X| \, |\mathcal{B}] I(\{E[|X| \, |\mathcal{B}] > a\}).$$

Take expectations. Writing $A := \{E[|X|\,|\mathcal{B}] \geq a\}$, the RHS gives $\int_A E[|X|\,|\mathcal{B}]dP$, and as A is $\mathcal{B}$-measurable, this is $\int_A E[|X|]dP$, by definition of conditional expectation. Splitting between $\{|X| \leq b\}$ and $\{|X| > b\}$, this is at most

$$P(E[|X|\,|\mathcal{B}] \geq a) + \int_{\{|X|>b\}} |X|dP.$$

But

$$P(E[|X|\,|\mathcal{B}] \geq a) \leq a^{-1}E[E[|X||\mathcal{B}]]$$

by Markov's Inequality, which is $a^{-1}E|X|$ by the Conditional Mean Formula. Combining,

$$\sup_{\mathcal{B}} \int_A E[|X|\,|\mathcal{B}]dP \leq \frac{b}{a}E|X| + \int_{\{|X|>b\}} |X|dP.$$

Take $b := \sqrt{a}$ and let $a \to \infty$: RHS $\to 0$ (as $X \in L_1$), so LHS $\to 0$. This says that $\{E[X|\mathcal{B}]\}$ is UI, as required. //

Theorem (Lévy). If $Y \in L_1$ and $(\mathcal{F}_n)$ is a filtration with $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then

$$E[Y|\mathcal{F}_n] \to E[Y|\mathcal{F}_\infty] \quad \text{a.s and in } L_1.$$

Proof. If $X_n := E[Y|\mathcal{F}_n]$, then X_n is a mg (w.r.t. $(\mathcal{F}_n)$), and is UI (by the UI Lemma). As $E[|X_n|] \leq E[|Y|] < \infty$, (X_n) is an L_1-bounded mg, so a.s. convergent (Doob's Mg Convergence Thm), to X_∞, say. Also X_n is L_1-convergent, by the Theorem (ii). It remains to show that $X_\infty = E[Y|\mathcal{F}_\infty]$. For $A \in \mathcal{F}_n$,

$$\int_A YdP = \int_A E[Y|\mathcal{F}_n]dP = \int_A X_n dP \to \int_A X_\infty dP,$$

by L_1-convergence. So

$$\int_A YdP = \int_A X_\infty dP,$$

for all $A \in \mathcal{F}_n$, for each n. As the $\mathcal{F}_n$ generate $\mathcal{F}_\infty$, this extends to $A \in \mathcal{F}_\infty$ (by a monotone class argument, or Carathéodory's Extension Theorem). As X_n is $\mathcal{F}_n$-measurable and $\mathcal{F}_n \subset \mathcal{F}_\infty$, X_n is $\mathcal{F}_\infty$-measurable, hence so is its limit X_∞. So

$$X_\infty = E[Y|\mathcal{F}_\infty],$$

by definition of conditional expectation. //

If the index set $\{1, 2, \ldots\}$ of the filtration $(\mathcal{F}_n)$ extends to $\{1, 2, \ldots, \infty\}$ so that $\{X_n : n = 1, 2, \ldots, \infty\}$ is a (sub-)mg w.r.t. this filtration, the (sub-)mg is called *closed*, with *closing* (or *last*) element X_∞.

Theorem. Let (X_n) be a UI submg. Then $\sup_n E[X_n^+] < \infty$, and X_n converges to a limit X_∞ a.s. and in L_1, which closes the submg.

Proof. By UI, $\sup E[|X_n|] < \infty$. So by Doob's Mg Convergence Thm, $X_n \to X_\infty$ a.s. Again by UI, $X_n \to X_\infty$ in L_1.

If $A_n \in \mathcal{F}_n$ and $k \geq n$, $\int_A X_n dP \leq \int_A X_k dP$ as (X_n) is a submg. Let $k \to \infty$: $X_k \to X_\infty$ in L_1 gives $\int_A X_n dP \leq \int_A X_\infty dP$. So by definition of conditional expectation, $X_n \leq E[X_\infty|\mathcal{F}_\infty]$. So X_∞ closes the submg. //

Theorem. X_n is a UI mg iff there exists $Y \in L_1$ with

$$X_n = E[Y|\mathcal{F}_n].$$

Then $X_n \to E[Y|\mathcal{F}_\infty]$ a.s. and in L_1.

Proof. If X is a UI mg, it is closed (by X_∞), by above, and then $X_n \to X_\infty$ a.s. and in L_1; take $Y := X_\infty$.

Conversely, given $Y \in L_1$ and $X_n := E[Y|\mathcal{F}_n]$, (X_n) is a mg, and is UI by above; the convergence follows by Lévy's result above. //

Corollary (UI Mg Convergence Theorem). For a mg $X = (X_n)$, the following are equivalent:
(i) X is UI;
(ii) X converges a.s. and in L_1 (to X_∞, say);
(iii) X is closed by a random variable Y: $X_n = E[Y|\mathcal{F}_n]$;
(iv) X is closed by its limit X_∞: $X_n = E[X_\infty|\mathcal{F}_n]$.

Proof. It remains to identify Y with the a.s. (or L_1) limit X_∞, which follows by uniqueness of limits. //

Note. 1. The UI mgs (also called *regular* mgs) are the 'nice' mgs. Note that all the randomness is in the closing rv $Y = X_\infty$. As time progresses, more of

Y is revealed as more information becomes available (progressive revelation, as in a 'striptease').
2. UI mgs are also common, and crucially important in Mathematical Finance. There, one does two things: (i) *discount* all asset prices (so as to work with real rather than nominal prices); (ii) change from the real-world probability measure P to an equivalent martingale measure Q (EMM, or *risk-neutral measure*) under which discounted asset prices $\tilde{S}_t$ become (Q)-mgs:

$$\tilde{S}_t = E_Q[\tilde{S}_T|\mathcal{F}_t]$$

(here $T < \infty$ is typically the expiry time of an option). See e.g. [BK], esp. Ch. 4.

Matters are simpler in the L_p case for $p \in (1, \infty)$. Call $X = (X_n)$ L_p-*bounded* if

$$\sup_n \|X_n\|_p < \infty$$

(so in particular each $X_n \in L_p$). We may take $p = 2$ for simplicity, and because of the link with Hilbert-space methods and the important *Kunita-Watanabe Inequalities.*

Theorem (L_p-Mg Theorem). If $p > 1$, an L_p-bounded mg X_n is UI, and converges to its limit X_∞ a.s. and in L_p.

Proof. First, X_n is UI: for, if $a > 0$,

$$a^{p-1} \int_{\{|X_n|>a\}} |X_n| dP \leq \int |X_n|^p dP.$$

So if $C := \sup_n \|X_n\|_p < \infty$,

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \leq C^p / a^{p-1} \to a \qquad (a \to \infty)$$

(as $p > 1$), so X_n is UI.

So (UI Mg Th.) $X_n = E[X_\infty|\mathcal{F}_n]$, where $X_n \to X_\infty$ a.s. and $X_\infty \in L_1$. So $|X_n|^p \to |X_\infty|^p$ a.s. By Fatou's Lemma

$$\int |X_\infty|^p dP \leq \liminf \int |X_n|^p dP \leq C^p < \infty,$$

so $X_\infty \in L_p$.

If X_∞ is bounded ($|X_\infty(\omega)| \leq a$ for all ω), then $X_n = E[X_\infty|\mathcal{F}_n]$ is also bounded by a. Then $|X_n - X_\infty|^p \leq 2a^p$, and $X_n \to X_\infty$ in L_p follows by dominated convergence.

In the general case, we use

$$X_\infty = (X_\infty \wedge a) + (X_\infty - a)^+$$

(check). Then

$$\|E[X_\infty|\mathcal{F}_n] - X_\infty\|_p \leq \|E[X_\infty \wedge a|\mathcal{F}_n] - X_\infty \wedge a\|_p + 2\|(X_\infty - a)^+\|_p$$

(as conditional expectations decrease L_p-norms. This is true for $p \geq 1$, but simpler for $p = 2$ – the only case we shall need – as then conditional expectation is a *projection*. We quote this – see e.g. [S], Ch. 22 (p = 2), 23 ($p \in [1, \infty]$).) By the bounded case, the first term on RHS $\to 0$ as $n \to \infty$. The second term $\to 0$ as $a \to \infty$ by dominated convergence (recall $X_\infty \in L_p$). So $X_n = E[X_\infty|\mathcal{F}_n] \to X_\infty$ in L_p as well as a.s. //

7. Martingales in continuous time

A stochastic process $X = (X(t))_{0 \leq t < \infty}$ is a *martingale* (mg) relative to $(\{\mathcal{F}_t\}, P)$ if

(i) X is adapted, and $E|X(t)| < \infty$ for all $\leq t < \infty$;

(ii) $E[X(t)|\mathcal{F}_s] = X(s)$ P- a.s. $(0 \leq s \leq t)$,

and similarly for submgs (with $\leq$ above)and supermgs (with $\geq$).

In continuous time there are regularization results, under which one can take $X(t)$ RCLL in t (basically $t \to EX(t)$ has to be right-continuous). Then the analogues of the results for discrete-time martingales hold true. Among the contrasts with the discrete case, we mention that the Doob-Meyer decomposition below, easy in discrete time, is a deep result in continuous time.

Interpretation.

Martingales model fair games. Submartingales model favourable games. Supermartingales model unfavourable games.

Martingales represent situations in which there is no drift, or tendency, though there may be lots of randomness. In the typical statistical situation where we have *data = signal + noise,* martingales are used to model the noise component. It is no surprise that we will be dealing constantly with such decompositions later (with 'semi-martingales').

Closed martingales.

As in discrete time, some martingales are of the form

$$X_t = E[X|\mathcal{F}_t] \qquad (t \geq 0)$$

for some integrable random variable X. Then X is said to *close* (X_t), which is called a *closed* (or *closable*) martingale, or a *regular* martingale, or a *UI* mg. As before, closure is equivalent to UI, and closed/UI martingales have specially good convergence properties:

$$X_t \to X_\infty \qquad (t \to \infty) \qquad \text{a.s. and in } L_1,$$

and then also

$$X_t = E[X_\infty | \mathcal{F}_t], \qquad a.s.$$

Doob-Meyer Decomposition.

One version in continuous time of the Doob decomposition in discrete time – called the Doob-Meyer (or the Meyer) decomposition – follows next but needs one more definition. A process X is called of class (D) if

$$\{X_\tau : \tau \text{ a finite stopping time}\}$$

is uniformly integrable. Then a (càdlàg, adapted) process Z is a submartingale of class (D) if and only if it has a decomposition

$$Z = Z_0 + M + A$$

with M a uniformly integrable martingale and A a predictable increasing process, both null at 0. This composition is unique.

Square-integrable Martingales.

For $M = (M_t)$ a martingale, write $M \in \mathcal{M}^2$ if M is L_2-bounded:

$$\sup_t E(M_t^2) < \infty,$$

and $M \in \mathcal{M}_0^2$ if further $M_0 = 0$. Write $c\mathcal{M}^2$, $c\mathcal{M}_0^2$ for the subclasses of continuous M.

As in discrete time, for $M \in \mathcal{M}^2, M$ is convergent:

$$M_t \to M_\infty \qquad \text{a.s. and in mean square}$$

for some random variable $M_\infty \in L_2$. One can recover M from M_∞ by

$$M_t = E[M_\infty | \mathcal{F}_t].$$

The bijection

$$M = (M_t) \leftrightarrow M_\infty$$

is in fact an isometry, and as $M_\infty \in L_2$, which is a Hilbert space, so too is $\mathcal{M}^2$.

Again, notice that all the randomness in the mg (M_t) is in the limit random variable M.