ma414l9.tex

Lecture 9. 8.3.2012

Example. We calculate $\int W(u)dW(u)$. We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} W(0) = 0 & \text{if} \quad 0 \le u \le t/n, \\ W(t/n) & \text{if} \quad t/n < u \le 2t/n, \\ \vdots & \vdots \\ W((n-1)t/n) & \text{if} \quad (n-1)t/n < u \le t. \end{cases}$$

By definition,

$$\int_0^t W(u)dW(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} W(kt/n)(W((k+1)t/n) - W(kt/n)).$$

Replacing $W(kt/n)$ by $\frac{1}{2}(W((k+1)t/n) + W(kt/n)) - \frac{1}{2}(W((k+1)t/n) - W(kt/n))$, the RHS is

$$\sum \frac{1}{2}(W((k+1)t/n) + W(kt/n)).(W((k+1)t/n) - W(kt/n))$$

$$-\sum \frac{1}{2}(W((k+1)t/n) - W(kt/n)).(W((k+1)t/n) - W(kt/n)).$$

The first sum is $\sum \frac{1}{2}(W((k+1)t/n)^2 - W(kt/n)^2)$, which telescopes (as a sum of differences) to $\frac{1}{2}W(t)^2$ ($W(0) = 0$). The second sum is $\frac{1}{2}\sum(W(k+1)t/n) - W(kt/n))^2$, an approximation to the quadratic variation of W on $[0, t]$, which tends to $\frac{1}{2}t$ by Lévy's theorem on the QV. Combining,

$$\int_0^t W(u)dW(u) = \frac{1}{2}W(t)^2 - \frac{1}{2}t.$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of W.

One can construct a closely analogous theory for stochastic integrals with the Brownian integrator W above replaced by a square-integrable martingale integrator M. The properties above hold, with the Lemma (i) replaced by

$$E[(\int_0^t X(u)dM(u))^2] = E[\int_0^t X(u)^2 d\langle M\rangle(u)].$$

The natural class of integrands X to use here is the class of predictable processes (a slight extension of left-continuity of sample paths).
Quadratic Variation, Quadratic Covariation.

We shall need to extend quadratic variation and quadratic covariation to stochastic integrals. The quadratic variation of $I_t(X) = \int_0^t X(u)dW(u)$ is $\int_0^t X(u)^2 du$. This is proved in the same way as the case $X \equiv 1$, that W has quadratic variation process t. More generally, if $Z(t) = \int_0^t X(u)dM(u)$ for a continuous martingale integrator M, then $\langle Z\rangle(t) = \int_0^t X^2(u)d\langle M\rangle(u)$. Similarly (or by polarization), if $Z_i(t) = \int_0^t X_i(u)dM_i(u)$ $(i = 1, 2)$, $\langle Z_1, Z_2\rangle(t) = \int_0^t X_1(u)X_2(u)d\langle M_1, M_2\rangle(u)$.
Semi-martingales.

It turns out that semi-martingales give the natural class of stochastic integrators: one can define the stochastic integral

$$\int_0^t H(u)dX(u) = \int_0^t H(u)dM(u) + \int_0^t H(u)dA(u)$$

for predictable integrands H (as above), and for semi-martingale integrators X – but for no larger class of integrators, if one is to preserve reasonable convergence and approximation properties for the operation of stochastic integration. For details, see e.g. [P].

With integrands as general as above, stochastic integrals are no longer martingales in general, but only *local martingales* (see e.g. [P]: martingales on each $[0, T_n]$, for some sequence of stopping times $T_n \uparrow \infty$). For our purposes, one loses little by thinking of bounded integrands (recall that we usually have a finite time horizon T, the expiry time of an option, and that bounded processes are locally integrable, but not integrable in general).

2. Itô's Lemma

Suppose that b is adapted and locally integrable (so $\int_0^t b(s)ds$ is defined as an ordinary integral, as in I.4), and σ is adapted and measurable with $\int_0^t E(\sigma(u)^2)du < \infty$ for all t (so $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral, as above). Then

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$ (which is often called an Itô process). It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \qquad X(0) = x_0. \tag{$*$}$$

Now suppose $f : \mathbf{R} \to \mathbf{R}$ is of class C^2. The question arises of giving a meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$, and finding it. Given a partition $\mathcal{P}$ of $[0,t]$, i.e. $0 = t_0 < t_1 < \ldots < t_n = t$, we can use Taylor's formula to obtain

$$f(X(t)) - f(X(0)) = \sum_{k=0}^{n-1} f(X(t_{k+1})) - f(X(t_k))$$

$$= \sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) + \frac{1}{2}\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2$$

with $0 < \theta_k < 1$. We know that $\sum(\Delta X(t_k))^2 \to \langle X\rangle(t)$ in probability (so, taking a subsequence, with probability one), and a little more effort gives

$$\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \to \int_0^t f''(X(u))d\langle X\rangle(u).$$

The first sum is easily recognized as an approximating sequence of a stochastic integral (compare the example above), giving

$$\sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \to \int_0^t f'(X(u))dX(u) :$$

Theorem (Basic Itô Formula). If X has stochastic differential given by $(*)$ and $f \in C^2$, then $f(X)$ has stochastic differential

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d\langle X\rangle(t),$$

or writing out the integrals,

$$f(X(t)) = f(x_0) + \int_0^t f'(X(u))dX(u) + \frac{1}{2}\int_0^t f''(X(u))d\langle X\rangle(u).$$

More generally, suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. By the Taylor expansion of a smooth function of several variables we get for t close to t_0 (we use subscripts to denote partial derivatives: $f_t := \partial f/\partial t$, $f_{tx} := \partial^2 f/\partial t\partial x$):

$$\begin{aligned} f(t, X(t)) &= f(t_0, X(t_0)) \\ &\quad +(t-t_0)f_t(t_0, X(t_0)) + (X(t) - X(t_0))f_x(t_0, X(t_0)) \\ &\quad +\tfrac{1}{2}(t-t_0)^2 f_{tt}(t_0, X(t_0)) + \tfrac{1}{2}(X(t) - X(t_0))^2 f_{xx}(t_0, X(t_0)) \\ &\quad +(t-t_0)(X(t) - X(t_0))f_{tx}(t_0, X(t_0)) + \ldots, \end{aligned}$$

which may be written symbolically as

$$df = f_t dt + f_x dX + \frac{1}{2} f_{tt}(dt)^2 + f_{tx} dt dX + \frac{1}{2} f_{xx}(dX)^2 + \ldots.$$

In this, we substitute $dX(t) = b(t)dt + \sigma(t)dW(t)$ from above, to obtain

$$\begin{aligned} df &= f_t dt + f_x(bdt + \sigma dW) \\ &\quad + \tfrac{1}{2} f_{tt}(dt)^2 + f_{tx} dt(bdt + \sigma dW) + \tfrac{1}{2} f_{xx}(bdt + \sigma dW)^2 + \ldots \end{aligned}$$

Now using the formal multiplication rules $dt \cdot dt = 0, dt \cdot dW = 0, dW \cdot dW = dt$ (which are just shorthand for the corresponding properties of the quadratic variations), we expand

$$(bdt + \sigma dW)^2 = \sigma^2 dt + 2b\sigma dt dW + b^2(dt)^2 = \sigma^2 dt + \text{higher-order terms}$$

to get finally

$$df = \left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW + \text{higher-order terms.}$$

As above, the higher-order terms are irrelevant, and summarizing, we obtain *Itô's lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem (Itô's Lemma). If $X(t)$ has stochastic differential given by $(*)$, then $f = f(t, X(t))$ has stochastic differential

$$df = \left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f = f_0 + \int_0^t (f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx}) dt + \int_0^t \sigma f_x dW. \qquad //$$

Corollary. $E(f(t, X(t))) = f_0 + \int_0^t E(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx}) dt$.

Proof. $\int_0^t \sigma f_2 dW$ is a stochastic integral, so a (local) martingale, so its expectation is constant ($= 0$, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's lemma really comes into its own in the more general setting of semi-martingales (of which X above is an important example). It says there that if X is a semi-martingale and f is a smooth function as above, then $f(t, X(t))$ is also a semi-martingale. The ordinary differential dt gives rise to the finite-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.
Itô Lemma in Higher Dimensions.

If $f(t, x_1, \ldots, x_d)$ is C^1 in its zeroth (time) argument t and C^2 in its remaining d space arguments x_i, and $M = (M_1, \ldots, M_d)$ is a continuous vector martingale, then (writing f_i, f_{ij} for the first partial derivatives of f with respect to its ith argument and the second partial derivatives with respect to the ith and jth arguments) $f(t, M(t))$ has stochastic differential

$$df(t, M(t)) = f_0(t, M(t))dt + \sum_{i=1}^{d} f_i(t, M(t))dM_i(t) + \frac{1}{2}\sum_{i,j=1}^{d} f_{ij}(t, M(t))d\langle M_i, M_j\rangle(t).$$

Application. The case $f(x) = x^2$ gives

$$W(t)^2 = W(0)^2 + \int_0^t 2W(u)dW(u) + \frac{1}{2}\int_0^t 2du,$$

which after rearranging is just our earlier example.

3. Geometric Brownian Motion

Now that we have both BM W and Itô's Lemma to hand, we can introduce the most important stochastic process for us, a relative of BM – *geometric* (or *exponential*, or *economic*) BM.

To model the stock-price evolution, we use the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \qquad S(0) > 0,$$

due to Itô in 1944. (Interpretation: the return dS/S over a short time-interval is the sum of the deterministic term μdt and the random term σdW.) This corrects Bachelier's earlier attempt of 1900 (he did not have the factor $S(t)$ on the right - missing the interpretation in terms of returns, and leading to negative stock prices!) Incidentally, Bachelier's work served as Itô's motivation in introducing Itô calculus. The mathematical importance of Itô's work was recognised early, and led on to the work of Doob in 1953 [D], Meyer

(1960s on) and many others. The economic importance of geometric Brownian motion was recognized by Paul A. Samuelson in his work from 1965 on, for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton, in work for which he was similarly honoured in 1997.

The differential equation above has the unique solution

$$S(t) = S(0) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma dW(t)\right\}.$$

For, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right) f, \qquad f_x = \sigma f, \qquad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \qquad (dx)^2 = dt.$$

Thus Itô's lemma gives

$$\begin{aligned} df(t,W(t)) &= f_t dt + f_x dW(t) + \tfrac{1}{2} f_{xx} (dW(t))^2 \\ &= f\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dW(t) + \tfrac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW(t)), \end{aligned}$$

so $f(t,W(t))$ is a solution of the stochastic differential equation, and the initial condition $f(0,W(0)) = S(0)$ as $W(0) = 0$, giving existence.

For uniqueness, we need the *stochastic* (or Doléans, or Doléans-Dade) *exponential* (below), giving $Y = \mathcal{E}(X) = \exp\{X - \frac{1}{2}\langle X\rangle\}$ (with X a continuous semi-martingale) as the unique solution to the stochastic differential equation

$$dY(t) = Y(t-)dX(t), \qquad Y(0) = 1.$$

(Incidentally, this is one of the few cases where a stochastic differential equation can be solved explicitly. Usually we must be content with an existence and uniqueness statement, and a numerical algorithm for calculating the solution.) Thus $S(t)$ above is the stochastic exponential of $\mu t + \sigma W(t)$, Brownian motion with mean (or drift) μ and variance (or volatility) σ^2. In particular,

$$\log S(t) = \log S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)$$

has a normal distribution. Thus $S(t)$ itself has a *lognormal* distribution. This geometric Brownian motion model, and the log-normal distribution that it entails, are the basis for the Black-Scholes model for stock-price dynamics in continuous time.

4. Stochastic Calculus for Black-Scholes Models; Girsanov's theorem

In this section we collect the main tools for the analysis of financial markets with uncertainty modelled by Brownian motions.

Consider first independent $N(0,1)$ random variables $Z_1, \ldots, Z_n$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\gamma = (\gamma_1, \ldots, \gamma_n)$, consider a new probability measure Q on $(\Omega, \mathcal{F})$ defined by

$$Q(d\omega) = \exp\left\{\sum_{i=1}^{n} \gamma_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2\right\} P(d\omega).$$

As $\exp\{.\} > 0$ and integrates to 1, as $\int \exp\{\gamma_i Z_i\} dP = \exp\{\frac{1}{2}\gamma_i^2\}$, this is a probability measure. It is also equivalent to P (has the same null sets), again as the exponential term is positive. Also

$$Q(Z_i \in dz_i, \quad i = 1, \ldots, n) = \exp\left\{\sum_{i=1}^{n} \gamma_i Z_i - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2\right\} P(Z_i \in dz_i,\ i = 1, \ldots, n)$$

$$= (2\pi)^{-n/2} \exp\left\{\sum_{i=1}^{n} \gamma_i z_i - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2 - \frac{1}{2}\sum_{i=1}^{n} z_i^2\right\} \prod_{i=1}^{n} dz_i$$

$$= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} (z_i - \gamma_i)^2\right\} dz_1 \ldots dz_n.$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\gamma_i, 1)$ under Q. Thus the effect of the *change of measure* $P \to Q$, from the original measure P to the *equivalent* measure Q, is to *change the mean*, from $0 = (0, \ldots, 0)$ to $\gamma = (\gamma_1, \ldots, \gamma_n)$.

This result extends to infinitely many dimensions. Let $W = (W_1, \ldots W_d)$ be a d-dimensional Brownian motion defined on a stochastic basis with the filtration satisfying the usual conditions. Let $(\gamma(t) : 0 \le t \le T)$ be a measurable, adapted d-dimensional process with $\int_0^T \gamma_i(t)^2 dt < \infty$ a.s., $i = 1, \ldots, d$, and define the process $(L(t) : 0 \le t \le T)$ by

$$L(t) = \exp\Big\{-\int_0^t \gamma(s)' dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)^2\| ds\Big\}.$$

Then L is continuous, and, being the stochastic exponential of $-\int_0^t \gamma(s)' dW(s)$, is a local martingale. Given sufficient integrability on the process γ, L will in fact be a (continuous) martingale. For this, *Novikov's condition* suffices:

$$E\left(\exp\left\{\frac{1}{2}\int_0^T \|\gamma(s)^2\| ds\right\}\right) < \infty.$$

We are now in the position to state a version of Girsanov's theorem, which is one of the main tools in studying continuous-time financial market models.

Theorem (Girsanov). Let γ be as above and satisfy Novikov's condition; let L be the corresponding continuous martingale. Define the processes $\tilde{W}_i$, $i = 1, \ldots, d$ by

$$\tilde{W}_i(t) := W_i(t) + \int_0^t \gamma_i(s) ds, \qquad (0 \leq t \leq T), \qquad i = 1, \ldots, d.$$

Then under the equivalent probability measure Q defined on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym derivative

$$\frac{dQ}{dP} = L(T),$$

the process $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_d)$ is d-dimensional Brownian motion.

In particular, for $\gamma(t)$ constant $(= \gamma)$, change of measure by introducing the Radon-Nikodym derivative $\exp\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\}$ corresponds to a change of drift from c to $c - \gamma$. If $(\mathcal{F}_t)$ is the Brownian filtration (basically $\mathcal{F}_t = \sigma(W(s), 0 \leq s \leq t)$ slightly enlarged to satisfy the usual conditions) any pair of equivalent probability measures $Q \sim P$ on $\mathcal{F} = \mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = L(t)$$

with L defined as above.

Note. The main application of the Girsanov theorem in mathematical finance is the change of measure in the Black-Scholes model of a financial market to obtain the risk-neutral martingale measure, under which discounted asset prices give prices of derivatives (options etc.). The relevant mathematics needed includes the next result (Brownian Martingale Representation Theorem).