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MA414 STOCHASTIC ANALYSIS: MOCK EXAMINATION, 2011

Q1. On a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, if P is a probability measure, $X \in L_1(P)$ is a positive random variable, and

$$Q(A) := E[XI(A)]/E[X] \qquad (A \in \mathcal{A}),$$

show that
(i) Q is a probability measure on $(\Omega, \mathcal{A})$; **[2]**
(ii) Q is absolutely continuous w.r.t. P; **[2]**
(iii) the Radon-Nikodym derivative is $dQ/dP = X/E[X]$; **[2]**
(iv) denoting expectation under Q by E_Q, then for $Z \in L_1(P)$

$$E_Q[Z] = E[XZ]/E[X]. \qquad \textbf{[2]}$$

For a convex function ϕ, state without proof Jensen's Inequality. **[2]**

By applying Jensen's Inequality with $\phi(x) = x^p$ for $p > 1$ to Q with X^p in place of X and choosing $Z := Y/X^{p-1}$, or otherwise, obtain Hölder's Inequality

$$\|XY\|_1 \leq \|X\|_p.\|Y\|_q,$$

where $q > 1$ is the conjugate index to p:

$$\frac{1}{p} + \frac{1}{q} = 1. \qquad \textbf{[15]}$$

Q2. State Fatou's Lemma without proof. **[3]**

Given random variables $X, Y \geq 0$ with $Y \in L_p$ for $p > 1$, and such that

$$xP(X \geq x) \leq E[YI(X \geq x)] \qquad \text{for all } x \geq 0, \qquad (*)$$

show that:
(i) $X \in L_p$; **[8]**
(ii)

$$\|X\|_p \leq \frac{p}{p-1}\|Y\|_p. \qquad \textbf{[14]}$$

(You may quote Hölder's Inequality without proof.)

Q3. (i) State and prove Markov's Inequality. **[2,4]**
(ii) Show that convergence in L_1 implies convergence in probability. **[5]**
(iii) Show that conditional expectation is a contraction: if $\mathcal{C}$ is a sub-σ-field and $X \in L_1$, $E[|E[X|\mathcal{C}]|] \leq E[|X|]$. **[8]**
(iv) Define uniform integrability of random variables X_n. **[2]**
(v) If X_n is uniformly integrable and converges to X a.s., show that for $\mathcal{C}$ a sub-σ-field $E[X_n|\mathcal{C}] \to E[X|\mathcal{C}]$ in L_1 and in probability. **[4]**

Q4. (i) Define a *local martingale* $X = (X_t)$ and a *stopping time* τ. If X is a local martingale and τ is a stopping time, show that $Y = (Y_t)$, where $Y_t := X_{t \wedge \tau}$, is also a local martingale. **[2,2,4]**
(ii) Show that if X is a bounded continuous local martingale, then X is a martingale. **[5]**
(ii) Show that a non-negative local martingale is a supermartingale. **[6]**
(iv) Show that a non-negative local martingale $X = (X_t : 0 \leq t \leq T)$ with $E[X_T] = E[X_0]$ is a martingale. **[6]**

Q5. (i) For N Poisson distributed with parameter λ and $X_1, X_2, \ldots$ independent of each other and of N, each with distribution F with mean μ, variance σ^2 and characteristic function $\phi(t)$, show that the compound Poisson distribution of

$$Y := X_1 + \ldots + X_N$$

has characteristic function $\psi(t) = \exp\{-\lambda(1 - \phi(t))\}$, mean $\lambda\mu$ and variance $\lambda E[X^2]$. **[7, 5, 5]**
(ii) Obtain the mean and variance of Y also from the Conditional Mean Formula and the Conditional Variance Formula. **[4, 4]**

Q6. For $B = (B_t)$ Brownian motion and $M = (M_t)$, where

$$M_t := (B_t^2 - t)^2 - 4\int_0^t B_s^2 ds,$$

(i) find the stochastic differential of M. **[8]**
(ii) Hence or otherwise, express M as an Itô integral, and show that M is a continuous martingale starting at 0. **[5, 5]**
(iii) Find the quadratic variation $[M]_t$ of M_t. **[7]**

N. H. Bingham