ma414prob1.tex

MA414 PROBLEMS 1. 12.1.2012

Q1. Show that the *power set* $\mathcal{P}(\Omega)$ of all subsets of a set Ω forms a ring (the *Boolean ring*) with addition as symmetric difference Δ and multiplication as intersection $\cap$. Find the zero element 0 and the identity element 1. Show that $x^2 = x$ for each element x, and find the additive inverse of each element.

Q2. Show that the union of countably many countable sets is countable.

Q3. Show that the union of countably many null sets is null.

Q4. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$, show that
(i) if $A_n \uparrow$ i.e. $A_n \subset A_{n+1} \subset \ldots$), then

$$\mu(A_n) \uparrow \mu(\cup A_n)$$

('continuity of μ from below');
(ii) if $A_n \downarrow$ (i.e. $A_n \supset A_{n+1} \supset \ldots$) and some $\mu(A_N) < \infty$

$$\mu(A_n) \downarrow \mu(\cap A_n)$$

('continuity of μ from above').

Q5. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$, show that
(i)

$$\mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n);$$

(ii) if $\mu(\cup_{k \geq N} A_k) < \infty$ for some N,

$$\mu(\text{limsup } A_n) \geq \text{limsup } \mu(A_n).$$

Q6. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$ is a convergent sequence of sets with $\mu(\cup A_n) < \infty$, show that

$$\mu(\lim A_n) = \lim \mu(A_n).$$

NHB