ma414prob3.tex

MA414 PROBLEMS 3. 30.1.2012

Q1 *Borel's normal Number Theorem (1909).* A number x is called *normal to base* 10 if the frequency of each of $0, 1, \ldots, 9$ in its decimal expansion is $1/10$; similarly for normal to base 2 in binary (0, 1 have frequency 1/2 each), and to base d $(0, 1, \ldots, d-1$ have frequency $1/d$ each). Show that almost all x are normal to all bases d simultaneously.

A number x is *strongly normal* to base d if for each k the frequencies of each of the d^k possible k-tuples is $1/d^k$, and similarly for the expansion shifted $1, 2, \ldots, k-1$ places to the right. Show that almost all numbers are strongly normal to all bases simultaneously.

Note. 1. *No* specific example of such a number x is known explicitly, though we know that *almost all* numbers are like this. This is a fine example of a non-constructive existence proof! The best known result is that Champernowne's numbe, 0.123456789101112...9899100101... is strongly normal to all bases that are powers of 10.

2. A variant of this is folklore statements such as that monkeys would eventually type the complete works of Shakespeare, etc.

3. The decimal expansion of π is known to billions of places, and passes all the standard statistical tests for randomness – but whether or not π is normal is unknown.

Q2 (Another partial converse to the First Borel-Cantelli Lemma). If $c > 0$ and each $P(A_n) \geq c$, show that $A := \limsup A_n$ has $P(A) \geq c$.

Q3. If $X \sim U(0,1)$ and $A_n := \{X < 1/n\}$, show that $\sum P(A_n) = \infty$ but $P(\limsup A_n) = 0$. Why does this not contradict the Second Borel-Cantelli Lemma?

Q4. Show that

$$1 + n + n^2/2! + \ldots + n^n/n! \sim \frac{1}{2}e^n \qquad (n \to \infty).$$

NHB