ma414prob5.tex

MA414 PROBLEMS 5. 9.2.2012

Q1 *(Doob's Submartingale Inequality).* For X a submg, $c \geq 0$, show that

$$cP(\max_{k=1,\ldots,n} X_k \geq c) \leq E[X_n I(\max_{k=1,\ldots,n} X_k \geq c)] \leq E[X_n].$$

Q2 *(Second Borel-Cantelli Lemma under Pairwise Independence).* Show that the Second Borel-Cantelli Lemma continues to hold with independence weakened to pairwise independence. [This result is due to Etemadi; there is a proof in [S], Th. 18.9, p.198-9. It uses Tchebycheff's inequality, and the fact that the Bernoulli distribution $B(p)$ with parameter $p \in [0,1]$ has mean p and variance $p(1-p)$ (or pq with $q := 1-p$)].

A function ϕ is called *convex* if $\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$ for all $\lambda \in [0,1], x, y$. We quote *Jensen's Inequality*: if X and $\phi(X)$ are integrable, then

$$E[\phi(X)] \leq \phi(E[X]),$$

and the *conditional Jensen inequality*: for $\mathcal{B}$ a σ-field,

$$E[\phi(X)|\mathcal{B}] \leq \phi(E[X|\mathcal{B}]).$$

Q3. (i) If $M = (M_t)$ is a mg, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submg.
(ii) If in (i) M is a submg, and ϕ is also non-decreasing on the range of M, show that again $\phi(M)$ is a submg.
(You may assume the conditional Jensen inequality.)

Q4. Deduce Kolmogorov's Inequality from Doob's Submg Inequality.

NHB