ma414prob6.tex

## MA414 PROBLEMS 6. 16.2.2012

Recall the following three fundamental convergence theorems.
*Lebesgue's Monotone Convergence Theorem.* If $X_n$ are non-negative random variables and $X_n$ increases to $X$, then $E[X_n]$ increases to $E[X]$.
*Fatou's Lemma.* If $X_n$ are integrable and bounded below, then $E[\liminf X_n] \leq \liminf E[X_n]$.
*Lebesgue's Dominated Convergence Theorem.* If $X_n \to X$ and $|X_n| \leq Y$ with $E[Y] < \infty$, then $E[X_n] \to E[X]$.
Prove the following conditional forms, where $\mathcal{C}$ denotes a $\sigma$-field.

Q1 (*Conditional monotone convergence*). If $X_n \geq 0$, $X_n \uparrow X$ and $X \in L_1$, show that

$$E[X_n|\mathcal{C}] \uparrow E[X|\mathcal{C}].$$

Q2 (*Conditional Fatou Lemma*). If $X_n \geq 0$, $X_n \in L_1$, show that

$$E[\liminf X_n|\mathcal{C}] \leq \liminf E[X_n|\mathcal{C}].$$

Q3 (*Conditional Dominated Convergence*). If $|X_n| \leq Y$, $Y \in L_1$, $X_n \to X$, show that

$$E[X_n|\mathcal{C}] \to E[X|\mathcal{C}].$$

For $p \geq 1$, call $q \geq 1$ the *conjugate index* to $p$ if

$$\frac{1}{p} + \frac{1}{q} = 1$$

(so $p = 1$ iff $q = \infty$). We write $L_p$ for the space of $p$th-power integrable functions $f$ – those with $\int |f|^p d\mu < \infty$ (or $E[|X|^p] < \infty$ for a probability measure). We quote (for proofs, see e.g. [S]):
*Hölder's inequality*: for $p > 1$, if $f \in L_p$, $g \in L_q$, then $fg \in L_1$, and

$$\int |fg| \leq (\int |f|^p)^{1/p} (\int |g|^q)^{1/q};$$

*Minkowski's inequality*: for $p \geq 1$, if $f, g \in L_p$, then $f + g \in L_p$, and

$$(\int |f+g|^p)^{1/p} \leq (\int |f|^p)^{1/p} + (\int |g|^p)^{1/p}.$$

We write $\|f\|_p := (\int |f|^p)^{1/p}$, called the *p-norm* of $f$. Then Minkowski's Inequality says that $L_p$ is a *normed* (vector) *space* under this norm, while Hölder's inequality says that

$$(f,g) := |\int fg| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q,$$

giving a *duality* between $L_p$ and $L_q$ for $p, q > 1$. Note that $p = q$ iff $p = q = 2$; then $L_2$ is called (a) *Hilbert space*, and the above gives an *inner product* on $L_2$. (Note that $L_2$ is the only *self-dual* $L_p$-space.)

Q4. Show that for $x, y > 0$, $p, q > 1$ conjugate indices,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Q5. *Conditional Hölder Inequality.*

If $p > 1$ with conjugate index $q$, $X \in L_p$, $Y \in L_q$, show that

$$|E[XY|\mathcal{C}]| \leq (E[|X|^p|\mathcal{C})^{1/p}.(E[|Y|^q|\mathcal{C})^{1/q}.$$

NHB