ma414prob8.tex

PROBLEMS 8. 1.3.2012

Q1 (*Brownian bridge*). Brownian bridge (or tied-down BM, or pinned BM) is the process B_0 defined by

$$B_0(t) := B_t - tB_1, \qquad 0 \le t \le 1.$$

Show that Brownian bridge is a continuous Gaussian process starting at 0 at time 0 and finishing at 0 at time 1. Find
(i) its covariance function;
(ii) its expansion in the Schauder functions.

Q2. Show (by time-inversion of BM, or otherwise) that

$$B_t/t \to 0 \qquad a.s. \qquad (t \downarrow 0).$$

Q3. Show that BM $B = (B_t(\omega)) = (B(t,\omega))$ is measurable (that is, measurable in (t,ω) w.r.t. $\lambda \times P$, the product measure of Lebesgue measure (for t) and the probability measure (for ω).

We quote two properties of BM.
1. *Law of the Ierated Logarithm (LIL).* $\limsup B_t/\sqrt{2t \log\log t} = +1$ a.s.
2. *Nowhere differentiability.* B_t is nowhere differentiable in t, a.s.

A set A is *perfect* if it is closed, and each of its points is a limit of points of A other than itself. A set A is *nowhere dense* if every open set contains an open set disjoint from A.

Q4. Show that the zero-set $Z := \{t : B_t = 0\}$ of BM is perfect, Lebesgue-null and nowhere dense.

Q5 (*Scheffé's Lemma*). If f_n, f are probability densities, and $f_n \to f$ a.e., show that (for Borel sets $B \in \mathcal{B}$)

$$\sup_{B \in \mathcal{B}} \left| \int_B f_n - \int_B f \right| \le \int |f_n - f| \to 0.$$

NHB