## PROBLEMS 9. 8.3.2012

For $B = (B_t)$ BM, write $B^* = (B_t^*)$, where $B_t^* := \sup_{[0,t]} B(.)$. By a result of Lévy, there exists a process, the *local time* $L = (L_t)$ of BM at 0, that grows only when BM is at 0, i.e. when $t \in Z$, for which

$$(B^* - B, B^*) =_d (|B|, L).$$

In particular,

$$B^* - B =_d |B|, \qquad B^* =_d L.$$

Also, $L$ grows only on $Z$, where $|B| = 0$, while $B^*$ grows only on the *record set* $R$, where $B_t^* - B_t = 0$ – i.e., where $B$ is at a current record high level. So

$$R =_d Z.$$

Q1. Write down as many properties of the random set $R$ as you can.

Q2. *First-passage time process of Brownian motion.* For $t \geq 0$ and $B$ BM, write

$$\tau_t := \min\{u : B_u \geq t\}$$

(or $\min\{u : B_u = t\}$ as BM is continuous). So $B(\tau_t) = t$.
(i) Show that for fixed $s \geq 0$, $M_t := \exp\{sB_t - \frac{1}{2}ts^2\}$ is a mg.
(ii) By considering the first-passage times of BM to levels $t$ and $t + u$, show that the process $\tau = (\tau_t)$ is a non-decreasing Lévy process (a subordinator).
(iii) By considering the bounded stopping times $T_n := \min(n, \tau_t)$ and Doob's Stopping Time Principle in continuous time (which you may quote) and letting $n \to \infty$, or otherwise, show that

$$E \exp\{-s\tau_t\} = e^{-t\sqrt{2s}}.$$

(iv) Show that for $c > 0$,

$$\tau_t =_d \tau_{ct}/c^2$$

(so $\tau$ is *stable* of index $1/2$ – the stable subordinator of index $1/2$.

Q3. Show that $\tau_1$ has density

$$f(x) := \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x^{3/2}} \cdot \exp\left(-\frac{1}{2x}\right).$$

[Let $f$ have Laplace transform $\phi(s) := \int_0^\infty e^{-sx} f(x) dx$. Find $\phi'(s)$, and show, by the change of variable

$$x = \frac{1}{2su}, \qquad \text{so} \qquad sx = \frac{1}{2u}, \qquad \frac{1}{2x} = su,$$

that $\phi$ satisfies the ODE $\phi'(s)/\phi(s) = -1/\sqrt{2s}.$]
(This result is due to Lévy; it is, with BM and the Cauchy distribution, one of the few explicit formulae for a stable density.)

Q4. If $X, X_1, \ldots, X_n$ are independent with the stable-1/2 density in Q2, show that $(X_1 + \ldots + X_n)/n^2 =_d X$. Why does this not contradict the SLLN?

NHB