ma414prob2.tex

MA414 SOLUTIONS 2. 26.1.2012

Q1.

$$\hat{\mu}_k := \overline{X^k} \to E[X^k] = \mu_k \quad a.s. \quad (n \to \infty).$$

The kth *central moment* is $\hat{\mu}_k^0 := \overline{(X - \overline{X})^k}$. Then

$$\begin{aligned}\hat{\mu}_k^0 := \overline{(X - \overline{X})^k} &= \overline{\textstyle\sum_0^k \binom{k}{i} X^i (-)^{k-i} (\overline{X})^{k-i}} \\ &= \textstyle\sum_0^k \binom{k}{i} (\overline{X^i})(-)^{k-i} (\overline{X})^{k-i}.\end{aligned}$$

By SLLN, as $n \to \infty$ this tends a.s. to

$$\textstyle\sum_0^k \binom{k}{i} E[X^i](-)^{k-i}[EX]^{k-i} = \sum_0^n E[(X - EX)^k] = \mu_k^0. \qquad //$$

Q2.

$$\log(1 + \mu t + \frac{1}{2}\mu_2 t^2 + \ldots) = \mu t + \frac{1}{2}\mu_2 t^2 + \ldots - \frac{1}{2}(\mu t + \ldots)^2 + \ldots$$

$$= \mu t + \frac{1}{2}(\mu_2 - \mu^2)t^2 + \ldots = \mu t + \frac{1}{2}\sigma^2 t^2 + \ldots.$$

Equating coefficients of t gives $\kappa_1 = \mu$. Centring at the mean multiplies the MGF $M(t)$ by $e^{-\mu t}$, so subtracts μt from the CGF, so leaves coefficients of powers of t^k unchanged for $k \geq 2$, so $\kappa_k = \kappa_{k,0}$ for $k \geq 2$, giving (i). Equating coefficients of t^2 gives $\kappa_2 = \sigma^2$, giving (ii). We can now take $\mu = 0$ w.l.o.g.:

$$M(t) = 1 + \frac{1}{2}\sigma^2 t^2 + \frac{1}{6}\mu_3^0 t^3 + \frac{1}{24}\mu_4^0 t^4 + \ldots$$

Take logs and use $\log(1 + x) = x - \frac{1}{2}x^2 + \ldots$:

$$K(t) = \log M(t) = \frac{1}{2}\sigma^2 t^2 + \frac{1}{6}\mu_3^0 t^3 + \frac{1}{24}\mu_4^0 t^4 + \ldots - \frac{1}{2}[\frac{1}{2}\sigma^2 t^2 \ldots]^2.$$

As $K(t) = \sum \kappa_k t^k / k!$, equating coefficients of t^3 gives $\kappa_3 = \mu_3^0$, which is (iii). Equating coefficients of t^4 gives $\kappa_4 = \mu_k^0 - 3\sigma^4$ (3 8s are $24 = 4!$), which is (iv). Finally, X is normal $N(\mu, \sigma)$ iff $M(t) = e^{\mu t + \sigma^2 t^2/2}$ iff $K(t) = \mu t + \sigma^2 t^2/2$ iff all cumulants higher than the second vanish.

Q3. Take $f(z) := e^{-z^2/2}$. This is entire (has no singularities). So for any contour γ, $\int_\gamma f = 0$, by Cauchy's Residue Theorem (or, use Cauchy's Theorem). Take γ the rectangle with vertices R, $R + iy$, $-R + iy$, $-R$, with sides γ_1 the interval $[-R, R]$, γ_2 the vertical line from R to $R + iy$, γ_3 the horizontal line from $R + iy$ to $-R + iy$, γ_4 the vertical line from $-R + iy$ to $-R$. So $\sum_1^4 \int_{\gamma_i} f = 0$. On γ_2, γ_4: $z = \pm R + iuy$ $(0 \leq u \leq 1)$,

$$f(z) = \exp\{-(\pm R + iuy)^2/2\} = e^{-R^2/2} e^{u^2 y^2/2} e^{\pm iRuy} \to 0 \qquad (R \to \infty),$$

as $|e^{\pm iRuy}| = 1$. So $\int_{\gamma_2} f \to 0$, $\int_{\gamma_4} f \to 0$ $(R \to \infty)$. Also $\int_{\gamma_1} f \to \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ as $R \to \infty$). Combining,

$$\int_{\gamma_3} f \to \int_{\infty}^{-\infty} e^{-x^2/2}.e^{y^2/2}.e^{-ixy} dx = -\sqrt{2\pi} \qquad (R \to \infty).$$

So (dividing by $\sqrt{2\pi}$ and by $e^{y^2/2}$, and reversing the direction of integration)

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{-ixy} dx = e^{-y^2/2}.$$

The RHS is real, so the LHS is real. Take complex conjugates:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{ixy} dx = e^{-y^2/2}.$$

This gives the characteristic function (CF) of the standard normal density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ (the CF is the *Fourier transform* of a probability density).

Q4. (i) If $F(t) := \int_0^\infty e^{-x} \cos xt dx$,

$$\begin{aligned}
F(t) &= \int_0^\infty e^{-x} \cos xt dx = -\int_0^\infty \cos xt de^{-x} \\
&= -[\cos xt.e^{-x}]_0^\infty + \int_0^\infty e^{-x}(-t \sin xt) dx \\
&= 1 = t \int_0^\infty \sin xt de^{-x} \\
&= 1 + t[\sin xt.e^{-x}]_0^\infty - t \int_0^\infty e^{-x}.t \cos xt dx \\
&= 1 - t^2 \int_0^\infty e^{-x} \cos xt dx = 1 - t^2 F(t) :
\end{aligned}$$

$$F(t)(1+t^2) = 1, \qquad F(t) = 1/(1+t^2).$$

Then

$$\int_{-\infty}^{\infty} e^{ixt}.\frac{1}{2}e^{-|x|}dx = \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx + i\int_{-\infty}^{\infty} \sin xt.\frac{1}{2}e^{-|x|}dx$$

$$= \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx = 1/(1+t^2),$$

by above (the second integral is zero: *odd* integrand, symmetric limits. The first integral is twice $\int_0^\infty$: *even* integrand, symmetric limits.

Thus the characteristic function of the *symmetric exponential* probability density $\frac{1}{2}e^{-|x|}$ is $1/(1+t^2)$.

(ii). Take $\epsilon > 0$. $f(z) = 1/(\pi(1+z^2))$ (to use Jordan's Lemma for $e^{itz}/(\pi(1+z^2)))$. The only singularity inside γ is at $y = i$, a simple pole.

$$Res_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi.2i} = \frac{-ie^{-t}}{2\pi}.$$

By Cauchy's Residue Theorem:

$$\int_\gamma f = 2\pi i.\left(\frac{-ie^{-t}}{2\pi}\right) = e^{-t}.$$

But

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad \text{(Jordan's Lemma)}.$$

This gives the result for $t > 0$. For $t = 0$, it is an arctan (or $\tan^{-1}$) integral. For $t < 0$: replace t by $-t$. //

Thus the CF of the symmetric Cauchy density $1/(\pi(1+x^2))$ is $e^{-|t|}$.

Q5. The similarity between (i) and (ii) of Q4 is an instance of the *Fourier Integral Theorem*: under suitable conditions, doing the Fourier transform twice gets back to where we started, apart from (a) e^{ixt} first time, but e^{-ixt} the second time; (b) a factor $1/2\pi$. (There are various formulations, depending on the class of function and type of integral – see a good book on Analysis, or a book on Fourier Analysis .) In Q3, the function $e^{-x^2/2}$ is its own Fourier transform (to within the constant factor $1/\sqrt{2\pi}$).

NHB