ma414soln3.tex

MA414 SOLUTIONS 3. 7.2.2011

Q1. We are using 'almost all' for numbers on the line; this means under Lebesgue measure (unless otherwise stated). We are talking about decimal expansions; this relates to the fractional part, which is in $[0,1]$; Lebesgue measure on $[0,1]$ is the uniform distribution. One can check that the decimal expansion coefficients of $X \sim U(0,1)$ are independent, uniformly distributed on the set $\{0,1,\ldots,d-1\}$, and conversely that if we have $e_1,\ldots,\epsilon_n,\ldots$ independent and uniform on this set, then

$$X := \sum_{n=1}^{\infty} \epsilon_n/d^n \sim U(0,1).$$

Normality to base d then follows by the SLLN applied to the uniform distribution on $\{0,1,\ldots,d-1\}$. The exceptional set is a null set for each d. The union of these countably many null sets for $d = 1,2,\ldots$ is also null, which gives almost all numbers normal to all bases simultaneously.

A similar argument applies to the d^k k-tuples for each d, and their k shifts. The union of the exceptional null sets over all d and k is still null, giving strong normality to all bases simultaneously, a.e.

Q2. As $P(A_n) \geq c > 0$, $P(A_n^c) \leq 1-c < 1$. So for each n,

$$P(\cap_{k=n}^{\infty} A_k^c) \leq P(A_n^c) \leq 1-c.$$

Now

$$A := \limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \qquad A^c = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^c = \lim_n \cap_{k=n}^{\infty} A_k^c,$$

as the sets $\cap_{k=n}^{\infty} A_k^c$ are increasing in n. So

$$P(A^c) = \lim_n P(\cap_{k=n}^{\infty} A_k^c) \leq 1-c : \qquad P(A) \geq c > 0.$$

Q3. $P(A_n) = 1/n$, and the harmonic series $\sum 1/n$ diverges. But $\{\limsup A_n\} = \{A_n \ i.o.\} = \emptyset$, so has probability 0. This does not contradict the Second Borel-Cantelli Lemma as the A_n are not independent (indeed, they are heavily dependent).

Q4. The Poisson distribution $P(\lambda)$ with parameter λ has mean λ and variance λ. Also, the sum of n independent $P(\lambda)$ variables is $P(n\lambda)$ (one can check these statements from the MGF of $P(\lambda)$, $M(t) = \exp\{-\lambda(1-e^t)\}$). Use the CLT with X_i Poisson $P(1)$; then $\mu = 1$, $\sigma = 1$, $S_n \sim P(n)$. Take $x = 0$ in CLT:

$$P(S_n - n)/\sqrt{n} \le 0) \to \Phi(0) = 1/2.$$

This says that

$$P(S_n - n \le 0) = P(S_n \le n) = \sum_{k=0}^{n} e^{-n} n^k / k! \to 1/2,$$

which is the conclusion required on multiplying by e^n.

NHB