ma414soln4.tex

MA414 SOLUTIONS 4. 6.2.2012

Q1. A is a tail event (convergence is unaffected by changes in finitely many terms), so has probability 0 or 1 by Kolmogorov's Zero-One Law.

Q2. (i) $f(x) = \sqrt{2/\pi}.x^{-3/2}.\exp(-1/(2x))$,

$$\phi(s) = \sqrt{2/\pi}\int_0^\infty \exp\Big(-sx - \frac{1}{2x}\Big)x^{-3/2}dx,$$

$$-\phi'(s) = \sqrt{2/\pi}\int_0^\infty \exp\Big(-sx - \frac{1}{2x}\Big)dx/\sqrt{x}.$$

Make the substitution from x to u, where $su := 1/(2x)$. Then $sx = 1/(2u)$ (so the substitution interchanges the two terms in the exponential), and $dx/\sqrt{x} = du/\sqrt{u}.(-)/\sqrt{2s}$. This gives

$$-\phi'(s) = \frac{1}{\sqrt{2s}}.\phi(s).$$

This is a first-order linear differential equation for $\phi(s)$, with initial condition $\phi(0) = 1$ (as f is a probability density). Solving, $\phi(s) = e^{-\sqrt{2s}}$, as required.
(ii) $X + Y$ has LST

$$E[e^{-s(X+Y)}] = E[e^{-sX}.e^{-sY}].$$

As X, Y are independent, so are e^{-sX}, e^{-sY}, so the RHS is

$$E[e^{-sX}].E[e^{-sY}] = \phi(s)\psi(s),$$

as required.
(iii) By (ii), $X_1 + \ldots + X_n$ has LST the nth power of the LST of X_1, namely $e^{-n\sqrt{2s}}$. Replacing s by s/n^2, $X_1 + \ldots + x_n)/n^2$ has LST $e^{-\sqrt{2s}}$, the LST of X_1, so $(X_1 + \ldots + X_n)/n^2 =_d X_1$.
(iv) The mean of X_1 is infinite (for large x, $f(x) \sim c/x^{3/2}$, so the integral for $EX = \int xf(x)dx$ diverges at $+\infty$). So SLLN does not apply.

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