ma414soln6.tex

MA414 SOLUTIONS 6. 23.2.2012

Q1 (*Conditional monotone convergence*). As integration is order-preserving, so is conditional expectation. So

$$E[X_n|\mathcal{B}] \le E[X_{n+1}|\mathcal{B}] \le E[X|\mathcal{B}].$$

So $E[X_n|\mathcal{B}]$ is increasing, and bounded above, so has a limit; as $E[X_n|\mathcal{B}]$ is $\mathcal{B}$-measurable, so is the limit. We need to show that this limit is $E[X|\mathcal{B}]$. Now for any $B \in \mathcal{B}$,

$$\begin{aligned}
\int_B \lim E[X_n|\mathcal{B}]dP &= \lim \int_B E[X_n|\mathcal{B}]dP \qquad \text{(Monotone Convergence)} \\
&= \lim \int_B X_n dP \qquad \text{(definition of conditional expectation)} \\
&= \int_B X dP \qquad \text{(Monotone Convergence)} \\
&= \int_B E[X|\mathcal{B}]dP \qquad \text{(definition of conditional expectation).}
\end{aligned}$$

As this holds for each $B \in \mathcal{B}$, $\lim E[X_n|\mathcal{B}] = E[X|\mathcal{B}]$ follows.

Q2 (*Conditional Fatour lemma*). Choose any $B \in \mathcal{B}$. By Fatou's Lemma applied to $X_n.I_B$,

$$\int_B \liminf X_n dP = \int \liminf I_B.X_n dP \le \liminf \int I_B.X_n dP = \liminf \int_B X_n dP.$$

The extreme left and extreme right here and the definition of conditional expectation give

$$\int_B E[\liminf X_n|\mathcal{B}]dP \le \liminf \int_B E[X_n|\mathcal{B}]dP.$$

As this holds for each $B \in \mathcal{B}$,

$$E[\liminf X_n|\mathcal{B}] \le \liminf E[X_n|\mathcal{B}].$$

Q3 (*Conditional dominated convergence*). Choose $B \in \mathcal{B}$. By dominated convergence applied to $X_n I_B$,

$$\int_B X_n dP \to \int_B X dP.$$

By definition of conditional expectation, this says

$$\int_B E[X_n|\mathcal{B}]dP \to \int_B E[X|\mathcal{B}]dP.$$

As this holds for all $B \in \mathcal{B}$,

$$E[X_n|\mathcal{B}] \to E[X|\mathcal{B}].$$

Q4. For $x, y > 0$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}. \tag{$*$}$$

One can check this by finding the maximum of the convex function $\phi(x) := xy - x^p/p$ (y fixed): $\phi'(x) = 0$ where $y - x^{p-1} - 0$, $x = y^{1/(p-1)}$. There,

$$\phi(x) = y^{\frac{1}{p-1}+1} - \frac{y^{\frac{p}{p-1}}}{p} = y^{\frac{p}{p-1}}(1 - \frac{1}{p}) = y^{\frac{p}{p-1}}/\frac{p}{p-1} = y^q/q,$$

a maximum (check).

Q5 (*Conditional Hölder Inequality*). By assumption, $f^p \in L^1$, $g^q \in L^1$, so $fg \in L^1$ by $(*)$. So all three conditional expectations are defined. By $(*)$,

$$\frac{|f|}{(E[|f|^p|\mathcal{B}])^{1/p}}.\frac{|g|}{(E[|g|^q|\mathcal{B}])^{1/q}} \leq \frac{|f|^p}{pE[|f|^p|\mathcal{B}]} + \frac{|g|^q}{qE[|f|^p|\mathcal{B}]}$$

on the set B where both denominators on LHS are positive. This set B is in $\mathcal{B}$, so we can take $E[.|\mathcal{B}]$ above, to get

$$\frac{E[|fg||\mathcal{B}]}{(E[|f|^p|\mathcal{B}])^{1/p}.(E[|g|^q|\mathcal{B}])^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

on B. As $|E[fg|\mathcal{B}]| \leq E[|fg||\mathcal{B}]$, this gives the result on B.

The set $B_1 := \{E[|f|^p|\mathcal{B}] = 0\} \in \mathcal{B}$. So by definition of conditional expectation,

$$\int_{B_1} |f|^p dP = \int_{B_1} E[|f|^p|\mathcal{B}]dP = 0,$$

so $f = 0$ a.s. on B_1; similarly, $g = 0$ a.s. on the corresponding set B_2 involving g. But since $B_1 \cup B_2 = B^c$, $fg = 0$ on B^c. So the result holds on B^c, so on $\Omega = B \cup B^c$. //

NHB