ma414soln7.tex

MA414 SOLUTIONS 7. 1.3.2012

Q1 (*Conditional Minkowski Inequality*). This follows just as Minkowski's Inequality follows from Hölder's Inequality (for this see e.g. [S], Th. 12.2 and Cor. 12.4), but with the conditional form of Hölder's Inequality in place of the unconditional one.

Q2. Map the points $n = 1, 2, 3, \dots$ on the half-line onto the points (r, s) in the first quadrant $(r, s = 1, 2, 3, \dots)$ by 'diagonal sweep':

$1 \to (1,1); 2 \to (2,1), 3 \to (1,2); 4 \to (3,1), 5 \to (2,2), 6 \to (1,3); 7 \to (4,1), \dots$

(draw a picture! This is the procedure used by Cantor in 1873 to show that the rationals are countable, but used in the reverse direction.) Then for each $r = 1, 2, \dots$,

$$U_r := \sum_{s=1}^{\infty} \epsilon_{(r,s)}/2^s \sim U(0,1),$$

and the U_r are independent.

Q3. (i) $P(X \leq x) = P(\Phi^{-1}(U) \leq x) = P(U \leq \Phi(x)) = x$, as $U \sim U(0,1)$, so $X \sim N(0,1)$.
(ii) With U_n as in Q2, $X_n := \Phi^{-1}(U_n) \sim N(0,1)$ and are independent.
(iii) Then as in lectures, $B(t) := \sum_0^\infty \lambda_n \Delta_n(t) X_n$ is Brownian motion.
(iv) Splitting each U_n in (ii) into infinitely many independent $U(0,1)$s as in Q2, and using each to generate a BM as in (iii), we get infinitely many independent BMs.

Q4. For $s \leq t$,

$$E[B_t^2|\mathcal{F}_s] = E[(B_s + (B_t - B_s))^2|\mathcal{F}_s]$$
$$= B_s^2 + B_s E[(B_t - B_s)|\mathcal{F}_s] + E[(B_t - B_s)^2] = B_s^2 + 0 + (t - s),$$

or

$$E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s,$$

showing that $(B_t^2 - t)$ is a mg. Since B_t^2 is a submg (Problems 5 Q3: B a mg, x^2 convex) and t is increasing,

$$B_t^2 = [B_t^2 - t] + t$$

is the Doob-Meyer decomposition of the submg $B_t^2 - t$. This identifies t as the QV of BM.

NHB