ma414soln8.tex

SOLUTIONS 8. 8.3.2012

Q1 (*Brownian bridge).* As $B_0(t) := B_t - tB_1$, B_0 is Gaussian (it is obtained from the Gaussian process B, BM, by forming linear combinations, and linear combinations of multinormals are multinormal). It is continuous as BM is; it starts from 0 as BM does, and finishes at 0 from its definition.
(i) The covariance function is

$$\begin{aligned} cov(B_0(s), B_0(t)] = E[(B_0(s)B_0(t)] &= E[(B_s - sB_1)(B_t - tB_1)] \\ &= E[B_sB_t - tB_sB_1 - sB_tB_1 + stB_1^2] \\ &= \min(s,t) - st - st + st \\ &= \min(s,t) - st \qquad (0 \leq s,t \leq 1) \end{aligned}$$

(as BM has covariance $\min(s,t)$ and $min(s,1) = s$, etc.).
(ii) Since BM is

$$B_t = \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) X_n$$

with Δ_n the Schauder functions and X_n independent $N(0,1)$: recall $\Delta_0(t) = t$, $\Delta_n(1) = 0$ for $n \geq 1$, and $B_1 \sim N(0,1)$. So putting $t = 1$ above gives $B_1 = X_0$. So B_0 is the sum of the remaining terms:

$$B_0(t) = \sum_{n=1}^{\infty} \lambda_n \Delta_n(t) X_n.$$

Q2. As in lectures, X_t (defined by time-inversion for $t > 0$) has the same covariance as BM. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.

Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \to 0$ as $t \to 0$. This says that

$$tB(1/t) \to 0 \qquad (t \to 0),$$

which is

$$B(t)/t \to 0 \qquad (t \to \infty),$$

as required.

Q3. Measurability of BM follows from the Schauder expansion. The partial sums are measurable in (t, ω) (recall the $\Delta_n(t)$ are continuous, and $X_n = X_n(\omega)$), and limits of measurable functions are measurable.

Q4. By LIL, $\limsup_{t\to\infty} B_t = +\infty$ a.s., and similarly (or by symmetry), $\liminf_{t\to\infty} B_t = -\infty$ a.s. So by continuity of BM, there must be arbitrarily large zeros of BM: the zero-set Z is unbounded, a.s. Then time-inversion (as in Q2) shows that there are zeros $t_n \downarrow 0$, a.s. – the zero at the starting-point $t = 0$ is followed by *infinitely many* zeros at *positive* times. Using the Strong Markov Property: *any* zero of BM must be a limit-point of zeros from the right. So any zero is a limit of zeros other than itself: Z is closed (by continuity of BM), and has no isolated points: Z is a *perfect set.*

With λ Lebesgue measure, we now evaluate $(\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\})$ in two ways, by Fubini's Theorem (which we can use, because $B_t(\omega) = B(t, \omega)$ is measurable). First,

$$(\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\}) = \int_0^\infty P(B_t = 0)dt = \int_0^\infty 0dt = 0.$$

Next,

$$(\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\}) = \int_\Omega \lambda(\{t : B_t(\omega) = 0\})dP = E[\lambda(Z)].$$

Combining, $E[\lambda(Z)] = 0$. But as $\lambda(Z) \geq 0$, this says $\lambda(Z) = 0$ a.s.: Z is a.s. *Lebesgue-null.*

As B_t is nowhere differentiable (given), B_t cannot vanish throughout any interval I (or it would have derivative 0 there). So by continuity, any interval I contains a subinterval J on which B_t is non-zero, i.e. J does not meet Z. So Z is *nowhere dense.*

Q5 (*Scheffé's Lemma).* $|\int_B f_n - \int_B f| = |\int_B (f_n - f| \leq \int_B |f_n - f|$. Taking sups over B proves the inequality. Next, with $a \wedge b := \min(a, b)$, $|f_n - f| = f_n + f - 2f_n \wedge f$ (check). Integrate: $\int f_n = 1$, $\int f = 1$ as these are densities. As $0 \leq f_n \wedge f \leq f$, integrable, dominated convergence gives $\int f_n \wedge f \to \int f = 1$. So the integral of RHS $\to$ 1+1-2 = 0. So the integral of LHS $\to$ 0 also. //

NHB