spexam2010.tex

## STOCHASTIC PROCESSES: EXAMINATION 2010

Answer five questions out of six; 20 marks per question.

Q1. State and prove Lebesgue's monotone convergence theorem. [**3, 7**]

Recall that the Gamma function and the Riemann zeta function are defined for real $s$ by

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx \quad (s > 0), \qquad \zeta(s) := \sum_{n=1}^\infty 1/n^s \quad (s > 1)$$

respectively. Show that [**10**]

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s) \qquad (s > 1).$$

Q2. By dyadic expansion, or otherwise, show how to identify the following two random experiments: [**8**]
(a) sampling a random number uniformly from $[0, 1]$;
(b) tossing a fair coin infinitely often independently.

Given one simulation from the uniform distribution on $[0, 1]$, show how to simulate the following:
(i) infinitely many independent uniform random variables; [**3**]
(ii) infinitely many independent standard normal random variables; [**3**]
(iii) Brownian motion; [**3**]
(iv) infinitely many independent Brownian motions. [**3**]

Q3. Define infinite divisibility, and state without proof the Lévy-Khintchine formula. [**2, 3**]

The *Cauchy density* is defined by $f(x) := 1/(\pi(1 + x^2))$.
(i) Show that it has characteristic function $\phi(t) = e^{-|t|}$. [**5**]
(ii) Deduce that it is infinitely divisible. [**2**]
(iii) If $X = (X_t)$ is the corresponding Lévy process (the *Cauchy process*), show that the Lévy measure $\mu$ of $X$ has density $1/(\pi|x|^2)$ (you may assume that $\int_0^\infty ((\sin\ x)/x) dx = \pi/2$). [**6**]

(iv) If $X_1, X_2, \ldots$ are independent with the Cauchy distribution, show that $(X_1 + \ldots + X_n)/n$ and $X_1$ have the same distribution. Why does this not contradict the Strong Law of Large Numbers? **[2]**

Q4. (i) For $B = (B_t) = (B(t))$ standard Brownian motion, define

$$X_t := tB(1/t) \qquad (t \neq 0).$$

Show that $X = (X_t)$ is again standard Brownian motion. **[6]**
(ii) Hence or otherwise, show that

$$B_t/t \to 0 \qquad a.s. \qquad (t \to \infty). \qquad \textbf{[6]}$$

(iii) Define the *Brownian bridge* $B_0$ by $B_0(t) := B(t) - tB_1$. Find the expansion of $B_0$ in terms of the Schauder functions $\Delta_n(t)$. **[8]**

Q5. (i) For $N$ Poisson distributed with parameter $\lambda$ and $X_1, X_2, \ldots$ independent of each other and of $N$, each with distribution $F$ with mean $\mu$, variance $\sigma^2$ and characteristic function $\phi(t)$, show that the compound Poisson distribution of

$$Y := X_1 + \ldots + X_N$$

has characteristic function $\psi(t) = \exp\{-\lambda(1 - \phi(t))\}$, mean $\lambda\mu$ and variance $\lambda E[X^2]$. **[6, 4, 4]**
(ii) Obtain the mean and variance of $Y$ also from the Conditional Mean Formula and the Conditional Variance Formula. **[3, 3]**

Q6. For $B = (B_t)$ Brownian motion and $M = (M_t)$, where

$$M_t := (B_t^2 - t)^2 - 4\int_0^t B_s^2 ds,$$

(i) find the stochastic differential of $M$. **[6]**
(ii) Hence or otherwise, express $M$ as an Itô integral, and show that $M$ is a continuous martingale starting at 0. **[4, 4]**
(iii) Find the quadratic variation $[M]_t$ of $M_t$. **[6]**

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