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STOCHASTIC PROCESSES: EXAMINATION SOLUTIONS, 12.12.2010

Q1. **Theorem (Lebesgue's monotone convergence theorem, 1902).** If f_n are non-negative measurable functions, $f_n \uparrow f$, then

$$\int f_n d\mu \uparrow \int f d\mu$$

(with both sides finite if $f \in L(\mu)$ and the RHS infinite otherwise).

Proof. For each n, choose f_{nk} simple increasing to f_n as $k \to \infty$. Then put $g_k := \max_{n \leq k} f_{nk}$. Then the g_k are increasing (with k), simple and non-negative, so

$$g_k \uparrow g \qquad (k \to \infty)$$

with g non-negative and measurable (as each g_k is). But for $n \leq k$

$$f_{nk} \leq g_k \leq f_k \leq f.$$

So letting $k \to \infty$,

$$f_n \leq g \leq f.$$

Letting $n \to \infty$, $f = g$. As the integral is order-preserving, by above

$$\int f_{nk} d\mu \leq \int g_k d\mu \leq \int f_k d\mu \qquad (n \leq k).$$

Let $k \to \infty$: by definition of the integral (via simple approximations),

$$\int f_n d\mu \leq \int g d\mu = \int f d\mu \leq \lim_{k \to \infty} \int f_k d\mu$$

(as $g = f$). Let $n \to \infty$:

$$\lim_{n \to \infty} \int f_n d\mu \leq \int f d\mu \leq \lim_{k \to \infty} \int f_k d\mu.$$

As the two extremes are equal, these are equalities, proving the result. //

For $s > 1$,

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \int_0^\infty x^{s-1}.e^{-x}.(1-e^{-x})^{-1} dx = \int_0^\infty x^{s-1}.e^{-x}.\sum_0^\infty e^{-nx} dx = \int_0^\infty x^{s-1}.\sum_1^\infty e^{-nx} dx.$$

Replacing the infinite sum on RHS by $lim_{n\to\infty} \sum_1^n ... = \lim_n f_n$, say, the f_n are increasing since each summand is positive. So by monotone convergence we may interchange limit and integral on the RHS, to get $\sum_1^\infty \int_0^\infty x^{s-1} e^{-nx} dx$. Replacing nx by x, this is

$$\sum_1^\infty \int_0^\infty x^{s-1} e^{-x} dx / n^s = \int_0^\infty x^{s-1} e^{-x} dx. \sum_1^\infty 1/n^s = \Gamma(s).\zeta(s). \quad //$$

Q2. Take the Lebesgue probability space $([0,1], \mathcal{L}, \mu)$ modelling the uniform distribution $U[0,1]$ on the unit interval (probability = length). For a random variable $X \sim U[0,1]$, take its dyadic expansion $X = \sum_1^\infty \epsilon_n/2^n$. Thus $\epsilon_1 = 0$ iff $X \in [0,1/2)$, 1 iff $X \in [1/2,1)$ (or $[1/2,1]$: we can omit 1, as it carries 0 probability). If $\epsilon_1, \ldots, \epsilon_{n-1}$ are already defined, on the dyadic intervals $[k/2^{n-1}, (k+1)/2^{n-1})$, split each interval into two halves: $\epsilon_n = 0$ on the left half, 1 on the right half. This construction shows that $\epsilon_1, \ldots, \epsilon_n$ are independent, coin-tossing random variables (Bernoulli with parameter 1/2: take values 0, 1 with probability 1/2 each), for each n. So the ϵ_n are independent coin-tosses. Conversely, given ϵ_n independent coin tosses, form $X := \sum_1^\infty \epsilon_n/2^n$. Then $X_n := \sum_1^n \epsilon_k/2^k \to X$ a.s. The distribution function of X_n has jumps $1/2^n$ at $k/2^n$, $k = 0, 1, \ldots, 2^n - 1$. This 'saw-tooth jump function' converges to x on $[0,1]$, the distribution function of $U[0,1]$. So $X \sim U[0,1]$. So if $X = \sum_1^\infty \epsilon_n/2^n$, $X \sim U[0,1]$ iff ϵ_n are independent coin tosses – the Lebesgue probability space models *both* (a) length on the unit interval *and* (b) infinitely many independent coin tosses.

(i) From the given $U[0,1]$, we get by dyadic expansion as above a sequence of independent coin-tosses ϵ_n. Rearrange these into a two-suffix array ϵ_{jk} (as with Cantor's proof of 1873 that the rationals are countable). The ϵ_{jk} are all independent, so the $X_j := \sum \epsilon_{jk}/2^k$ are independent, and $U[0,1]$ by above. So from *one* $U(0,1)$, we get in this way *infinitely many copies*.
(ii) If F is a distribution function (right-continuous; increasing from 0 at $-\infty$ to 1 at ∞), define its (left-continuous) inverse function by $F^{-1}(t) := \inf\{F(x) \geq t\}$ for $0 < t < 1$. Then if $U \sim U[0,1]$, $X := F^{-1}(U) \sim F$. For, $\{X \leq x\} = \{F^{-1}(U) \leq x\} = \{U \leq F(x)\}$, which has probability $F(x)$ as U is uniform. By this *probability integral transformation* we can pass from generating copies from the uniform distribution (say by Monte Carlo simulation) to generating copies from the distribution F, in particular, standard normals. Hence by (i) above we can then generate infinitely many independent standard normals.
(iii) We can hence simulate a Brownian motion $B = (B_t)$ from $B_t = \sum_0^\infty \lambda_n Z_n \Delta_n(t)$, with Z_n independent standard normals, $\Delta_n(t)$ the Schauder functions and λ_n suitable normalising constants.
(iv) Similarly, using (ii) rather than (i), we may simulate infinitely many independent Brownian motions.

Q3. A distribution is *infinitely divisible* (id) iff, for each $n = 1, 2, \ldots$, it is the n-fold convolution of a probability distribution – equivalently, if its CF is the nth power of the CF of a probability distribution.

The *Lévy-Khintchine formula* states that a probability distribution is id iff its CF has the form $\exp\{-\Psi(u)\}$, where

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (e^{iux} - 1 - I(|x| < 1)\mu(dx),$$

(a is real, $\sigma \geq 0$ and the Lévy measure μ satisfies $\int \min(1, |x|^2)\mu(dx) < \infty$).
(i) $\phi(t) = \int_{-\infty}^{\infty} e^{itx}/(\pi(1 + x^2))dx$. Take γ the semicircle in the upper half-plane on base $[-R, R]$, $t > 0$, and consider $f(z) := e^{itz}/(\pi(1+z^2))$. The only singularity inside γ is at $y = i$, a simple pole.

$$Res_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi.2i} = \frac{-ie^{-t}}{2\pi}.$$

By Cauchy's Residue Theorem:

$$\int_\gamma f = 2\pi i.\left(\frac{-ie^{-t}}{2\pi}\right) = e^{-t}.$$

But

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad \text{(Jordan's Lemma)}.$$

This gives the result for $t > 0$. For $t = 0$, it is an arctan (or $\tan^{-1}$) integral. For $t < 0$: replace t by $-t$. //

Thus the CF of the symmetric Cauchy density $1/(\pi(1 + x^2))$ is $e^{-|t|}$.
(ii) This is id, as $e^{-|t|} = [e^{-|t|/n}]^n$ for each n, and each [.] is a CF.
(iii) Substituting $\mu(dx) = 1/(\pi|x|^2)dx$ above gives $\Psi(u)$ as the sum of two integrals, I_1 over $(-1, 1)$ and I_2 over its complement. In I_1, the $\pm iux$ terms over $(-1, 0)$ and $(0, 1)$ cancel; we can then combine I_1 and I_2 to get

$$\Psi(u) = \frac{2}{\pi}\int_0^\infty (\cos\ ux - 1)dx/x^2.$$

This gives $\Psi'(u) = -(2/\pi)\int_0^\infty \sin ux\, dx/x = -(2/\pi)\int_0^\infty \sin\ t\, dt/t = -(2/\pi).\pi/2 = -1$. So $\Psi(u) = -u$ for $u > 0$. So $\Psi(u) = -|u|$. //

For X_i independent Cauchy, $(X_1 + \ldots + X_n)/n$ has CF $[e^{-|t|/n}]^n = e^{-|t|}$, the CF of X_1. So $(X_1 + \ldots + X_n)/n =_d X_1$. This does not contradict the SLLN: it does not apply, as the mean of X_i is undefined.

Q4. (i) For $t \neq 0$, X is Gaussian with zero mean (as B is), and continuous (again, as B is). The covariance of B is $\min(s,t)$. The covariance of X is

$$\begin{aligned} cov(X_s, X_t) &= cov(sB(1/s), tB(1/t)) \\ &= E[sB(1/s).tB(1/t)] \\ &= st.E[B(1/s)B(1/t)] \\ &= st.cov(B(1/s), B(1/t)) \\ &= st.\min(1/s, 1/t) = \min(t,s) = \min(s,t). \end{aligned}$$

This is the same covariance as Brownian motion. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.

(ii) Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \to 0$ as $t \to 0$. This says that

$$tB(1/t) \to 0 \qquad (t \to 0),$$

which is

$$B(t)/t \to 0 \qquad (t \to \infty),$$

as required.

By construction, Brownian motion is given by its expansion

$$B_t = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t),$$

where the Z_n are independent standard normal random variables, the $\Delta_n(t)$ are the Schauder functions and the λ_n are normalising constants. Now $\Delta_n(0) = 0$ for $n \geq 1$, while $\Delta_0(t) = t$, so $\Delta_0(1) = 1$. Also $\lambda_0 = 1$. Putting $t = 1$, $B_1 = Z_0$. So Brownian bridge is

$$B_0(t) := B(t) - tB(1) = B(t) - tZ_0 :$$

the expansion of Brownian bridge in the Schauder functions is

$$B_0(t) = \sum_{n=1}^{\infty} \lambda_n Z_n \Delta_n(t).$$

Q5. (i)

$$\begin{aligned}\psi(t) = E[e^{itY}] &= E[\exp\{it(X_1 + \ldots + X_N)\}] \\ &= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n].P(N = n) \\ &= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1 + \ldots + X_n)\}] \\ &= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n \\ &= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\ &= \exp\{-\lambda(1 - \phi(t))\}.\end{aligned}$$

Differentiate:

$$\psi'(t) = \psi(t).\lambda\phi'(t),$$
$$\psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives $EY = \lambda\mu$. Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2].$$

(ii) Given N, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As N is Poisson with parameter λ, N has mean λ and variance λ. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2].$$

Q6. (i). Write $f(B,t) := (B^2 - t)^2$. By Itô's formula,

$$df = f_B dB + f_t dt + \frac{1}{2}[f_{BB}(dB)^2 + 2f_{Bt} dB dt + f_{tt}(dt)^2].$$

In the [...] on RHS, $(dB)^2 = dt$, $dBdt = 0$, $(dt)^2 = 0$. Also

$$f_B = 2.2B(B^2 - t), \quad f_t = -2(B^2 - t), \quad f_{BB} = 4(B^2 - t) + 4B.2B = 12B^2 - 4t.$$

So

$$df = 4B(B^2 - t)dB - 2(B^2 - t)dt + (6B^2 - 2t)dt = 4B(B^2 - t)dB + 4B^2 dt.$$

As $M = f - 4\int_0^t B_s^2 ds$, the stochastic differential of M is

$$dM = df - 4B_t^2 dt = 4B(B^2 - t)dB.$$

(ii) So integrating, M is the Itô integral

$$M_t = 4\int_0^t B_s(B_s^2 - s)dB_s.$$

The Itô integral on the RHS is a continuous local martingale starting from 0. Now $B_t =_d t^{1/2}.Z$ where Z is $N(0,1)$. As Z has all moments finite, each $E[B_t^n]$ is a polynomial in t. So the integrand $h = h(B_t, t)$ on RHS satisfies the integrability condition $\int_0^t E[h_s^2]ds < \infty$ for all t. So the RHS is a (true) continuous mg starting from 0.
(iii). With $[M] = ([M_t])$ the quadratic variation of M,

$$d[M]_t = (dM)_t^2; \qquad dM_t = 4B_t(B_t^2 - t)dB_t.$$

So

$$d[M]_t = 16B_t^2(B_t^2 - t)^2(dB_t)^2 = 16B_t^2(B_t^2 - t)^2 dt :$$

$$[M]_t = 16\int_0^t B_s^2(B_s^2 - s)^2 ds.$$

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