spl1.tex
Lecture 1. 11.10.2010.

I. MEASURE THEORY AND INTEGRATION

1. Length, Area and Volume.

On the real line, for $a \leq b$, the *length* of the interval $[a, b]$ is (defined to be) $b - a$, and similarly for (a, b), $[a, b)$, $(a, b]$. In the plane, the *area* of a rectangle is (defined to be) the product of the (lengths of the) base and height. In three dimensions, the *volume* of a cuboid (3-dimensional analogue of a rectangle) is (defined to be) the product of the (lengths of the) three perpendicular sides (length x breadth x height). Similarly for d dimensions.

The length (or area, or volume) of a set which is the disjoint union of two sets above is (defined to be) the sum of the lengths (or areas, or volumes). Similarly for the disjoint union of a finite number n of such sets. This is the property of *finite additivity*. This is all very obvious, classical, and pre-1900.

What about sets that are *infinite* disjoint unions of sets above? It is quite reasonable to define the length etc., in this case as the sum of the lengths etc. – though now the series may diverge to $+\infty$ (not a problem – even in the above, we may have $a = -\infty$ and/or $b = +\infty$: lines, and half-lines, have infinite length; planes and half-planes have infinite area, etc.) This is *countable additivity*, or *σ-additivity*. This is not obvious, and it dates from the 1902 thesis of Henri LEBESGUE (1875-1941):
H. Lebesgue: Intégrale, longueur, aire. *Annali di Mat.* **7** (1902), 231-259.

Before proceeding, we ask for *which sets* can we calculate the length/area/volume? We recall some classical results on area and volume known to the ancient Greeks.
Circle.

Defining π by the perimeter of a circle of radius r being $2\pi r$ (and taking for granted that the ratio of perimeter to radius is the same for all circles!), the area A of a circle is πr^2.
First proof (Greeks). Divide the circle into a large even number $2n$ of equal sectors (360, say). Stack the sectors in a pile, of base r, with the odd-numbered vertices on the left and the evens on the right. The resulting shape is approximately rectangular; by symmetry, half the perimeter, i.e. πr, is on the left. So by the area of a rectangle, the area is approximately $r \times \pi r = \pi r^2$. The approximation (equivalent to $\sin\,\theta \sim \theta$ for θ small) can be made arbitrarily accurate by taking n large enough. So $A = \pi r^2$. //

Second proof (polar coordinates). The element of area in plane polar coordinates is

$$dA = dr.rd\theta = rdrd\theta.$$

Integrating over r from 0 to r and θ from 0 to 2π gives A $= 2\pi.r^2/2 = \pi r^2$. //

Ellipse.

The area of an ellipse with semi-axes a, b is $A = \pi ab$.

Proof (cartesian coordinates). Were the ellipse a circle ($a = b$), we would have the result by above. Suppose w.l.o.g. that $a \geq b$. Use cartesians, in which the element of area is

$$dA = dx.dy.$$

Squash the x-axis in the ratio b/a. This makes the ellipse a circle of radius b, so area πb^2. It also reduces the area of $dA = dx.dy$ to $dA = dxdy.b/a$. To get the area of the ellipse, we have to unsquash – i.e. to increase πb^2 in the ratio a/b, giving area $A = \pi ab$. //

Observe that we have now used both of the plane coordinate systems in common use. We have also exhausted the 'nice' examples, where we can find areas by exploiting the geometry of the figure. For general figures, there is only one method available: superimpose a suitably fine sheet of graph paper over the figure, and count squares inside it (including a correction for squares meeting the edge); then take finer and finer sheets of graph paper. This procedure will work for figures with nice regular boundaries, but we must expect it to fail for irregular boundaries, where the figure is 'all edge and no middle'.

These elementary thoughts strongly suggest that we cannot define area for general sets in the plane, only for suitably nice ones. This is indeed true, and similarly for length and volume in one and three dimensions (we used two dimensions above so that we can draw pictures).

The upshot is that we need a mathematical theory for measuring length, area and volume. This exists, and is called Measure Theory; it dates back to Lebesgue.

It turns out that the mathematics needed to handle length, area and volume actually works much more generally. We can also use it to handle gravitational mass (in Celestial Mechanics, following Newton's Inverse Square Law of Gravity in his *Principia* of 1687). We can also use it to handle electrostatic charge (which, unlike length, area, volume and mass, can be negative). Crucially, we can also use it to handle *probability*.

Perimeter and area of a circle.

If r increases to $r + dr$, the perimeter L increases to $2\pi(r + dr)$ and the area A to $\pi(r + dr)^2$. So $dA = 2\pi r dr + \pi(dr)^2 = 2\pi r dr = L dr$, to first order. So $dA/dr = L$, $A = \int L dr$. Thus either of the formulae for L and A follows from the other. Geometrically, this says that the annular strip dA can be straightened out to a rectangle of sides $L = 2\pi r$ and dr without changing its area (to first order).

Volume of a sphere.

We use spherical polar coordinates (what else?!) (r, θ, ϕ), where $\theta \in [0, \pi]$ is the colatitude and $\phi \in [0, 2\pi]$ the longitude. The element of volume is

$$dV = r^2 \sin\theta dr d\theta d\phi.$$

So

$$V = \int\int\int dV = \int_0^r r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = r^3/3.2\pi.[-\cos\theta]_0^\pi$$

$$= 2\pi r^3/3.(-)[(-1) - 1] = 4\pi r^3/3.$$

Surface area of a sphere.

For fixed radius r, the element of area is $dA = r^2 \sin\theta d\theta d\phi$, so

$$A = \int\int dA = r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = r^2.2.2\pi = 4\pi r^2.$$

Again, note that $dV/dr = A$, $V = \int_0^r dA$: if r is increased by dr, the extra volume dV is to first order Adr (think of melting of polar ice distributing itself over the surface of the oceans to increase sea level).

Volume of a pyramid, cone etc..

For a pyramid with a general (not necessarily square) base, or a cone with a (not necessarily circular) base, of base area B and height h, dividing up the volume into similar horizontal slices and integrating vertically gives

$$V = \int dV = \int_0^h (y/h)^2 B.dy = B.h^3/3.h^{-2} : \qquad V = Bh/3.$$

(Please continue this list with your own favourite examples.)

All this was known to the ancient Greeks, essentially by these methods. The Greeks had integration, which they called the 'method of exhaustion'

(Eudoxus (of Cnidus, c.410 - c.355 BC); in Euclid's Elements, Book X, c. 300BC; developed by Archimedes (c.287 - 212 BC)), but not explicitly differentiation, still less a second derivative – which is why Newton's Laws of Motion, and Law of Gravity, had to wait two millennia to launch the Scientific Revolution.

What had to wait even longer – till Lebesgue – was a systematic investigation of *which* sets have a length/area/volume. By above, we must expect that not all sets do! Indeed, a 'typical' set does not; those which do are reasonably regular or nice (technically: measurable). We turn to all this next.