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Lecture 18. 19.11.2010

Stopping Times and Optional Stopping

Recall (L17) that a random variable τ taking values in $\{0, 1, 2, \ldots; +\infty\}$ is called a *stopping time* if

$$\{\tau \leq n\} = \{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \qquad \forall\, n \leq \infty.$$

From $\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\}$ and $\{\tau \leq n\} = \bigcup_{k \leq n} \{\tau = k\}$, we see the equivalent characterization

$$\{\tau = n\} \in \mathcal{F}_n \qquad \forall\, n \leq \infty.$$

Call a stopping time τ *bounded* if there is a constant K such that $P(\tau \leq K) = 1$. (Since $\tau(\omega) \leq K$ for some constant K and all $\omega \in \Omega \setminus N$ with $P(N) = 0$ all identities hold true except on a null set, i.e. a.s.)

Example. Suppose (X_n) is an adapted process and we are interested in the time of first entry of X into a Borel set B (typically one might have $B = [c, \infty)$):

$$\tau = \inf\{n \geq 0 : X_n \in B\}.$$

Now $\{\tau \leq n\} = \bigcup_{k \leq n} \{X_k \in B\} \in \mathcal{F}_n$ and $\tau = \infty$ if X never enters B. Thus τ is a stopping time. Intuitively, think of τ as a time at which you decide to quit a gambling game: whether or not you quit at time n depends only on the history up to and including time n – NOT the future. Thus stopping times model gambling and other situations where there is no foreknowledge, or prescience of the future; in particular, in the financial context, where there is no insider trading. Furthermore since a gambler cannot cheat the system the expectation of his hypothetical fortune (playing with unit stake) should equal his initial fortune.

Theorem (Doob's Stopping-time Principle. Let τ be a bounded stopping time and $X = (X_n)$ a martingale. Then X_τ is integrable, and

$$E(X_\tau) = E(X_0).$$

Proof. Assume $\tau(\omega) \leq K$ for all ω, where we can take K to be an integer and write

$$X_{\tau(\omega)}(\omega) = \sum_{k=0}^{\infty} X_k(\omega) I(\tau(\omega) = k) = \sum_{k=0}^{K} X_k(\omega) I(\tau(\omega) = k).$$

Then

$$\begin{aligned}
E(X_\tau) &= E[\sum_{k=0}^{K} X_k I(\tau = k)] \qquad \text{(by the decomposition above)} \\
&= \sum_{k=0}^{K} E[X_k I(\tau = k)] \qquad \text{(linearity of } E\text{)} \\
&= \textstyle\sum_{k=0}^{K} E[E(X_K|\mathcal{F}_k) I(\tau = k)] \qquad (X \text{ a mg, } \{\tau = k\} \in \mathcal{F}_k \text{)} \\
&= \sum_{k=0}^{K} E[X_K I(\tau = k)] \qquad \text{(defn. of conditional expectation)} \\
&= E[X_K \sum_{k=0}^{K} I(\tau = k)] \qquad \text{(linearity of } E\text{)} \\
&= E[X_K] \qquad \text{(the indicators sum to 1)} \\
&= E[X_0] \qquad (X \text{ a mg}) \qquad //.
\end{aligned}$$

The stopping time principle holds also true if $X = (X_n)$ is a supermg; then the conclusion is

$$EX_\tau \leq EX_0.$$

Also, alternative conditions such as
(i) $X = (X_n)$ is bounded ($|X_n|(\omega) \leq L$ for some L and all n, ω);
(ii) $E\tau < \infty$ and $(X_n - X_{n-1})$ is bounded; suffice for the proof of the stopping time principle.

The stopping time principle is important in many areas, such as sequential analysis in statistics.

We now wish to create the concept of the σ-algebra of events observable up to a stopping time τ, in analogy to the σ-algebra $\mathcal{F}_n$ which represents the events observable up to time n.
Definition. Let τ be a stopping time. The *stopping time* $\sigma-$*algebra* $\mathcal{F}_\tau$ is defined to be

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq n\} \in \mathcal{F}_n, \text{ for all } n\}.$$

Proposition. For τ a stopping time, $\mathcal{F}_\tau$ is a $\sigma-$algebra.

Proof. We simply have to check the defining properties. Clearly $\Omega, \emptyset$ are in $\mathcal{F}_\tau$. Also for $A \in \mathcal{F}_\tau$ we find

$$A^c \cap \{\tau \leq n\} = \{\tau \leq n\} \setminus (A \cap \{\tau \leq n\}) \in \mathcal{F}_n,$$

thus $A^c \in \mathcal{F}_\tau$. Finally, for a family $A_i \in \mathcal{F}_\tau,\ i = 1, 2, \ldots$ we have

$$\left(\bigcup_{i=1}^{\infty} A_i\right) \cap \{\tau \leq n\} = \bigcup_{i=1}^{\infty} (A_i \cap \{\tau \leq n\}) \in \mathcal{F}_n,$$

showing $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\tau$. //

One can show similarly that for σ, τ stopping times with $\sigma \leq \tau$, $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Similarly, for any adapted sequence of random variables $X = (X_n)$ and a.s. finite stopping time τ, define

$$X_\tau := \sum_{n=0}^{\infty} X_n I(\tau = n).$$

Then X_τ is $\mathcal{F}_\tau$-measurable.

We are now in position to obtain an important extension of the Stopping-Time Principle.

Theorem (Doob's Optional-Sampling Theorem, OST. Let $X = (X_n)$ be a mg and σ, τ be bounded stopping times with $\sigma \leq \tau$. Then

$$E\left[X_\tau | \mathcal{F}_\sigma\right] = X_\sigma$$

and thus $E(X_\tau) = E(X_\sigma)$.

Proof. First observe that X_τ and X_σ are integrable (use the sum representation and the fact that τ is bounded by an integer K) and X_σ is $\mathcal{F}_\sigma$-measurable by above. So it only remains to prove that

$$E(I_A X_\tau) = E(I_A X_\sigma) \qquad \forall A \in \mathcal{F}_\sigma.$$

For any such fixed $A \in \mathcal{F}_\sigma$, define ρ by

$$\rho(\omega) = \sigma(\omega) I_A(\omega) + \tau(\omega) I_{A^c}(\omega).$$

Since

$$\{\rho \leq n\} = (A \cap \{\sigma \leq n\}) \cup (A^c \cap \{\tau \leq n\}) \in \mathcal{F}_n$$

ρ is a stopping time, and from $\rho \leq \tau$ we see that ρ is bounded. So the STP implies $E(X_\rho) = E(X_0) = E(X_\tau)$. But

$$\begin{aligned} E(X_\rho) &= E\left(X_\sigma I_A + X_\tau I_{A^c}\right), \\ E(X_\tau) &= E\left(X_\tau I_A + X_\tau I_{A^c}\right). \end{aligned}$$

So subtracting yields the result. //

We quote a further characterization of the martingale property. Let $X = (X_n)$ be an adapted sequence of random variables with $E(|X_n|) < \infty$ for all n and $E(X_\tau) = 0$ for all bounded stopping times τ. Then X is a martingale.

Write $X^\tau = (X_n^\tau)$ for the sequence $X = (X_n)$ stopped at time τ, where we define $X_n^\tau(\omega) := X_{\tau(\omega)\wedge n}(\omega)$. One can show
(i) If X is adapted and τ is a stopping time, then the stopped sequence X^τ is adapted.
(ii) If X is a martingale (super-, submartingale) and τ is a stopping time, X^τ is a martingale (super-, submartingale).

Examples and Applications.
1. Simple Random Walk. Recall the simple random walk: $S_n := \sum_{k=1}^n X_k$, where the X_n are independent tosses of a fair coin, taking values ± 1 with equal probability 1/2. Suppose we decide to bet until our net gain is first $+1$, then quit. Let τ be the time we quit; τ is a stopping time. The stopping time τ has been analyzed in detail; see e.g. [GS], S5.3, or Exercise 3.4. From this, note:
(i) $\tau < \infty$ a.s.: the gambler will certainly achieve a net gain of $+1$ eventually;
(ii) $E\tau = +\infty$: the mean waiting-time for this is infinity. Hence also:
(iii) No bound can be imposed on the gambler's maximum net loss before his net gain first becomes $+1$.

At first sight, this looks like a foolproof way to make money out of nothing: just bet until you get ahead (which happens eventually, by (i)), then quit. However, as a gambling strategy, this is hopelessly impractical: because of (ii), you need unlimited time, and because of (iii), you need unlimited capital – neither of which is realistic.

Notice that the Stopping-time Principle fails here: we start at zero, so $S_0 = 0$, $ES_0 = 0$; but $S_\tau = 1$, so $ES_\tau = 1$. This example shows two things:
1. Conditions are indeed needed here, or the conclusion may fail (none of the conditions in STP or the alternatives given are satisfied in this example).
2. Any practical gambling (or trading) strategy needs to have some integrability or boundedness restrictions to eliminate such theoretically possible but practically ridiculous cases.