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Because of the above corollary, we will not be able to define integrals with respect to Brownian motion by a path-by-path procedure (for BM the relevant convergence in the above results in fact takes place with probability one). However, turning to the class of square-integrable continuous martingales $c\mathcal{M}^2$ (continuous square-integrable martingales), we find that these processes have finite quadratic variation, but all variations of higher order are zero and, except for trivial cases, all variations of lower order are infinite with positive probability. So quadratic variation is indeed the right variation to study. Returning to Brownian motion, we observe that for $s < t$,

$$E(W(t)^2|\mathcal{F}_s) = E([W(s) + (W(t) - W(s))]^2$$
$$= W(s)^2 + 2W(s)E[(W(t) - W(s))|\mathcal{F}_s] + E[(W(t) - W(s))^2|\mathcal{F}_s]$$
$$= W(s)^2 + 0 + (t - s).$$

So $W(t)^2 - t$ is a martingale. This shows that the quadratic variation is the adapted increasing process in the Doob-Meyer decomposition of W^2 (recall that W^2 is a nonnegative submartingale and thus can be written as the sum of a martingale and an adapted increasing process). This result extends to the class $c\mathcal{M}^2$ (and indeed to the broader class of local martingales – below).

Theorem. A martingale $M \in c\mathcal{M}^2$ is of finite quadratic variation $\langle M \rangle$, and $\langle M \rangle$ is the unique continuous increasing adapted process vanishing at zero with $M^2 - \langle M \rangle$ a martingale.

The quadratic variation result above leads to Lévy's 1948 result, the martingale characterization of Brownian motion. Recall that $W(t)$ is a continuous martingale with respect to its natural filtration $(\mathcal{F}_t)$ and with quadratic variation t. There is a remarkable converse, due to Lévy:

Theorem (Martingale Characterization of BM). If M is any continuous, square-integrable (local) $(\mathcal{F}_t)$-martingale with $M(0) = 0$ and quadratic variation t, then M is an $(\mathcal{F}_t)$-Brownian motion.

Expressed differently this is:
If M is any continuous, square-integrable (local) $(\mathcal{F}_t)$-martingale with $M(0) =$

0 and $M(t)^2 - t$ a martingale, then M is an $(\mathcal{F}_t)$-Brownian motion.

In view of the fact that $\langle W \rangle(t) = t$, a further useful fact about Brownian motion may be guessed: If M is a continuous martingale then there exists a Brownian motion $W(t)$ such that $M(t) = W(\langle M \rangle(t))$, i.e. the martingale M can be transformed into a Brownian motion by a random time-change. These results already imply that Brownian motion is the fundamental continuous martingale.

Properties of Brownian Motion

Brownian Scaling. For any $c > 0$, write

$$W_c(t) := c^{-1} W(c^2 t), \qquad t \geq 0$$

with W BM. Then W_c is Gaussian, with mean 0, variance $c^{-2} \times c^2 t = t$ and covariance

$$cov(W_c(s), W_c(t)) = c^{-2} E(W_c(s), W_c(t)) = c^{-2} \min(c^2 s, c^2 t)$$
$$= \min(s, t) = cov(W(s), W(t)).$$

Also W_c has continuous paths, as W does. So W_c has all the properties of Brownian motion. So, W_c *is* Brownian motion. It is said to be derived from W by *Brownian scaling* with *scale-factor* $c > 0$. Since

$$(W(ut) : t \geq 0) = (\sqrt{u} W(t) : t \geq 0) \qquad \text{in law}, \forall u > 0,$$

W is called *self-similar* with *index* 1/2. Brownian motion is thus a *fractal.* A piece of Brownian path, looked at under a microscope, still looks Brownian, however much we 'zoom in and magnify'. Of course, the contrast with a function f with some smoothness is stark: a differentiable function begins to look straight under repeated zooming and magnification, because it has a tangent.

Time-Inversion. Write

$$X_t := t W(1/t).$$

Then X has mean 0 and covariance

$$cov(X_s, X_t) = st.cov(B(1/s), B(1/t)) = st. \min(1/s, 1/t) = \min(t, s) = \min(s, t).$$

Since X has continuous paths also, as above, X *is* Brownian motion. We say that X is obtained from W by *time-inversion.* This property is useful in transforming properties of BM 'in the large' ($t \to \infty$) to properties 'in the small', or local properties ($t \to 0$). For example, one can translate the law of the iterated logarithm (LIL) from global to local form.

Zero set Z of Brownian motion

This has many interesting properties – see the handout.

Parameters of Brownian Motion – Estimation and Hypothesis Testing.

If we form $\mu t + \sigma W_t$ – or replace $N(0,t)$ by $N(\mu t, \sigma t)$ in the definition of Brownian increments – we obtain a Lévy process that has continuous paths and Gaussian increments, called *Brownian motion* with *drift* μ and *diffusion coefficient* σ, $BM(\mu, \sigma)$, rather than *standard* Brownian motion $BM = BM(0,1)$ as above. By above, the quadratic variation of a segment of $BM(\mu, \sigma)$ path on the time-interval $[0,t]$ is $\sigma^2 t$, a.s. So, *if* we can observe a Brownian path completely over any time-interval however short, then *in principle* we can determine the diffusion coefficient σ *with probability one*. In particular, we can *distinguish* between two different σs – σ_1 and σ_2, say – with certainty. In technical language: the Wiener measures P_{*1} and P_{*2} representing these two Brownian motions with different σs on function space are *mutually singular*. By contrast, if the two σs are the same, the two measures are *mutually absolutely continuous*. We can then test a hypothesis $H_0 : \mu = \mu_0$ against an alternative hypothesis $H_1 : \mu = \mu_1$ by means of the appropriate *likelihood ratio* (LR). To find the form of the LR, one can use Girsanov's theorem. In practice, of course, we cannot observe a Brownian path *exactly* over a time-interval: there would be an infinite amount of information, and our ability to sample is finite. So one must use an appropriate discretization – and then we lose the ability to pick up the diffusion coefficient with certainty. Problems of this kind are not only of theoretical interest, but also important in practice. In mathematical finance, when the driving noise is modeled by Brownian motion, the diffusion coefficient is called the *volatility*, the parameter that describes how sensitive a stock-price is to price-sensitive information (or economic uncertainty, or driving noise). Volatility enters explicitly into the most famous formula of mathematical finance, the *Black-Scholes formula*. Volatility estimation is of major importance. So too is *volatility modeling*: alas, in real financial data the assumption of constant volatility is usually untenable for detailed modeling, and one resorts instead to more complicated models, say involving *stochastic volatility*.

6. Point Processes; Poisson processes

Point processes

Suppose that one is studying earthquakes, or volcanic eruptions. The events of interest are sudden isolated shocks, which occur at random instants, the history of which unfolds with time. Such situations occur in financial settings also: at the macro-economic level, the events might be stock-market crashes, devaluations etc. At the micro-economic level, they might be individual transactions. In other settings, the events might be the occurrence of telephone calls, insurance claims, accidents or admissions to hospital etc. The mathematical framework needed to handle such situations is that of point processes.

A *point process* is a stochastic process whose realizations are, not paths as above, but counting measures: random measures μ whose value on each interval I (or Borel set, more generally) is a non-negative integer $\mu(I)$. Often, each point may come labelled with some quantity (the size of the transaction, or of the earthquake on the Richter scale, for instance), giving what is called a marked point process. We turn below to the simplest and most fundamental point process, the Poisson process, and the simplest way to build it.

Stochastic processes with stationary independent increments are called *Lévy processes* (after the great French probabilist Paul Lévy in the 1930s. The two most basic prototypes of Lévy processes are Poisson processes and Brownian motion.

We include below a number of results without proof. For proofs and background, we refer to any good book on stochastic processes, e.g. [GS].