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Exponential Distribution

A random variable T is said to have an exponential distribution with rate λ , or $T = E(\lambda)$ if

$$P(T \le t) = 1 - e^{-\lambda t}$$
 for all $t \ge 0$.

Recall $E(T) = 1/\lambda$ and $var(T) = 1/\lambda^2$. Further important properties are: (i) Exponentially distributed random variables possess the 'lack of memory' property: P(T > s + t|T > t) = P(T > s).

(ii) Let T_1, T_2, \ldots, T_n be independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ resp. Then $\min\{T_1, T_2, \ldots, T_n\}$ is exponentially distributed with rate $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.

(iii) Let T_1, T_2, \ldots, T_n be independent exponentially distributed random variables with parameter λ . Then $G_n = T_1 + T_2 + \ldots + T_n$ has a $Gamma(n, \lambda)$ distribution. That is, its density is

$$P(G_n = t) = \lambda e^{-\lambda t} (\lambda t)^{n-1} / (n-1)! \quad \text{for } t \ge 0.$$

The Poisson Process

Definition. Let t_1, t_2, \ldots, t_n be independent exponential $E(\lambda)$ random variables. Let $T_n = t_1, + \ldots, + t_n$ for $n \ge 1$, $T_0 = 0$, and define $N(s) = \max\{n : T_n \le s\}$.

Interpretation: Think of t_i as the time between arrivals of events, then T_n is the arrival time of the *n*th event and N(s) the number of arrivals by time s. Then N(s) has a Poisson distribution with mean λs . The Poisson process can also be characterised via

Theorem. If $\{N(s), s \ge 0\}$ is a Poisson process, then (i) N(0) = 0, (ii) $N(t + s) - N(s) = Poisson(\lambda t)$, and (iii) N(t) has independent increments. Conversely, if (i),(ii) and (iii) hold, then $\{N(s), s \ge 0\}$ is a Poisson process. The above characterization can be used to extend the definition of the Poisson process to include time-dependent intensities. We say that $\{N(s), s \ge 0\}$ is a *Poisson process* with rate $\lambda(r)$ if (i) N(0) = 0,

(ii) N(t+s) - N(s) is Poisson with mean $\int_{s}^{t} \lambda(r) dr$, and

(iii) N(t) has independent increments.

Compound Poisson Processes

We now associate i.i.d. random variables Y_i with each arrival and consider

 $S(t) = Y_1 + \ldots + Y_{N(t)}, \qquad S(t) = 0 \text{ if } N(t) = 0.$

Theorem. Let (Y_i) be i.i.d. and N be an independent nonnegative integer random variable, and S as above.

(i) If $E(N) < \infty$, then $E(S) = EX(N).E(Y_1)$. (ii) If $E(N^2) < \infty$, then $var(S) = E(N).var(Y_1) + var(N)(E(Y_1))^2$. (iii) If N = N(t) is Poisson (λt) , then $var(S) = t\lambda(E(Y_1))^2$.

A typical application in the insurance context is a Poisson model of claim arrival with random claim sizes.

Renewal Processes

Suppose we use components – light-bulbs, say – whose lifetimes X_1, X_2, \ldots are independent, all with law F on $(0, \infty)$. The first component is installed new, used until failure, then replaced, and we continue in this way. Write

$$S_n := \sum_{1}^{n} X_i, \qquad N_t := \max\{k : S_k < t\}.$$

Then $N = (N_t : t \ge 0)$ is called the *renewal process* generated by F; it is a *counting process*, counting the number of failures seen by time t.

The law F has the *lack-of-memory property* iff the components show no aging – that is, if a component still in use behaves as if new. The condition for this is

$$P(X > s + t | X > s) = P(X > t) \qquad (s, t > 0),$$

or

$$P(X > s+t) = P(X > s)P(X > t).$$

Writing $\overline{F}(x) := 1 - F(x)$ $(x \ge 0)$ for the *tail* of F, this says that

$$\overline{F}(s+t) = \overline{F}(s)\overline{F}(t) \qquad (s,t \ge 0).$$

Obvious solutions are

$$\overline{F}(t) = e^{-\lambda t}, \qquad F(t) = 1 - e^{-\lambda t}$$

for some $\lambda > 0$ – the exponential law $E(\lambda)$. Now

$$f(s+t) = f(s)f(t) \qquad (s,t \ge 0)$$

is a 'functional equation' – the *Cauchy functional equation* – and it turns out that these are the *only* solutions, subject to minimal regularity (such as one-sided boundedness, as here – even on an interval of arbitrarily small length!).

So the exponential laws $E(\lambda)$ are characterized by the lack-of-memory property. Also, the lack-of-memory property corresponds in the renewal context to the Markov property. The renewal process generated by $E(\lambda)$ is called the Poisson (point) process with rate λ , $Ppp(\lambda)$. So: among renewal processes, the only Markov processes are the Poisson processes. We meet Lévy processes below: among renewal processes, the only Lévy processes are the Poisson processes.

It is the lack of memory property of the exponential distribution that (since the inter-arrival times of the Poisson process are exponentially distributed) makes the Poisson process the basic model for events occurring 'out of the blue'.

7. Lévy Processes

Distributions; The Lévy-Khintchine Formula

A distribution is *infinitely divisible* (id) if for each n its CF ϕ is the nth power ϕ_n^n of a CF ϕ_n . The class of infinitely divisible laws is written ID. The form of the general infinitely-divisible distribution was studied in the 1930s by several people (including Kolmogorov and de Finetti). The final result, due to Lévy and Khintchine, is expressed in CF language – indeed, cannot be expressed otherwise. The Lévy-Khintchine formula below is a static result; its dynamic counterpart involves Lévy processes (stochastic processes with stationary independent increments). We return to these in IV.6 [L30] in connection with stochastic calculus.

To describe the CF of the general i.d. law, we need three components: (i) a real *a* (called the *drift*, or deterministic drift), (ii) a non-negative σ (called the *diffusion coefficient*, or normal component, or Gaussian component), (iii) a (positive) measure μ on **R** (or **R** \ {0}) for which

$$\int_{-\infty}^{\infty} \min(1, |x|^2) \mu(dx) < \infty,$$

that is,

$$\int_{|x|<1} |x|^2 \mu(dx) < \infty, \qquad \int_{|x|\ge 1} \mu(dx) < \infty,$$

called the *Lévy measure*. The result is

Theorem (Lévy-Khintchine Formula). A function ϕ is the characteristic function of an infinitely divisible distribution iff it has the form

$$\phi(u) = \exp\left\{-\Psi(u)\right\} \qquad (u \in \mathbf{R}),$$

where

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI_{(-1,1)}(x)\mu(dx)) \qquad (L - K)$$

for some real $a, \sigma \geq 0$ and Lévy measure μ .

Examples. These include the normal, Poisson, compound Poisson and Cauchy laws (see below under 'Stability' for Cauchy).

The Central Limit Problem. In the CLT of II.7 [L12], we found that the limits we can get from an iid sequence by centring and scaling (subtracting means and dividing by variances there) were normal. The classical central limit problem generalizes this to sums $\Sigma_k X_{nk}$ from a 'triangular array' $(1 \le k \le k_n < \infty, n = 1, 2, ...; X_{nk}$ independent as k varies for fixed n). It turns out that the class of possible limit laws is exactly the class ID of infinitely divisible laws in (L - K).

Self-decomposability. Recall that if, in the central limit problem, we restrict from (two-suffix) triangular arrays (X_{nk}) to (one-suffix) sequences (X_n) , we come to a subclass of the infinite-divisible laws I, called the class of self-decomposable laws $SD : SD \subset I$.

Stability. If we further restrict to iid sequences X_n , we get the class S of stable laws: $S \subset SD \subset I$. To within location and scale, these are described by two parameters, the *index* $\alpha \in (0, 2]$ and the skewness parameter $\beta \in [-1, 1]$; $\alpha = 2$ gives the normal law and $\beta = 0$ gives symmetry. The (symmetric) Cauchy law is the case $\alpha = 1, \beta = 0$; density $1/(\pi(1 + x^2))$.