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Lecture 25. 6.12.2010

Lévy Processes

Suppose we have a process $X = (X_t : t \geq 0)$ that has stationary independent increments. Such a process is called a *Lévy process*, in honour of their creator, the great French probabilist Paul Lévy. Then for each $n = 1, 2, \ldots$,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \ldots + (X_t - X_{(n-1)t/n})$$

displays X_t as the sum of n independent (by independent increments), identically distributed (by stationary increments) random variables. Consequently, X_t is *infinitely divisible*, so its CF is given by the Lévy-Khintchine formula.

The prime example is: the Wiener process, or Brownian motion, is a Lévy process.

Poisson Processes. The increment $N_{t+u} - N_u$ $(t, u \geq 0)$ of a Poisson process is the number of failures in $(u, t+u]$ (in the language of renewal theory). By the lack-of-memory property of the exponential, this is independent of the failures in $[0, u]$, so the increments of N are *independent.* It is also identically distributed to the number of failures in $[0, t]$, so the increments of N are *stationary.* That is, N has stationary independent increments, so is a Lévy process: Poisson processes are Lévy processes.

We need an important property: two Poisson processes (on the same filtration) are independent iff they never jump together (a.s.).

The Poisson count in an interval of length t is Poisson $P(\lambda t)$ (where the rate λ is the parameter in the exponential $E(\lambda)$ of the renewal-theory viewpoint), and the Poisson counts of disjoint intervals are independent. This extends from intervals to Borel sets:
(i) For a Borel set B, the Poisson count in B is Poisson $P(\lambda|B|)$, where $|.|$ denotes Lebesgue measure; (ii) Poisson counts over disjoint Borel sets are independent.

Poisson (Random) Measures. If ν is a finite measure, call a random measure ϕ *Poisson* with *intensity* (or characteristic) *measure* ν if for each Borel set B, $\phi(B)$ has a Poisson distribution with parameter $\nu(B)$, and for $B_1, \ldots, B_n$ disjoint, $\phi(B_1), \ldots, \phi(B_n)$ are independent. One can extend to σ-finite measures ν: if (E_n) are disjoint with union $\mathbf{R}$ and each $\nu(E_n) < \infty$, construct

ϕ_n from ν restricted to E_n and write ϕ for $\sum \phi_n$.

Poisson Point Processes. With ν as above a (σ-finite) measure on $\mathbf{R}$, consider the product measure $\mu = \nu \times dt$ on $\mathbf{R} \times [0, \infty)$, and a Poisson measure ϕ on it with intensity μ. Then ϕ has the form

$$\phi = \sum_{t \geq 0} \delta_{(e(t),t)},$$

where the sum is *countable*. Thus ϕ is the sum of Dirac measures over 'Poisson points' $e(t)$ occurring at Poisson times t. Call $e = (e(t) : t \geq 0)$ a *Poisson point process* with *characteristic measure* ν,

$$e = Ppp(\nu).$$

For each Borel set B,

$$N(t, B) := \phi(B \times [0, t]) = card\{s \leq t : e(s) \in B\}$$

is the *counting process* of B – it counts the Poisson points in B – and is a Poisson process with rate (parameter) $\nu(B)$. All this reverses: starting with an $e = (e(t) : t \geq 0)$ whose counting processes over Borel sets B are Poisson $P(\nu(B))$, then – as no point can contribute to more than one count over disjoint sets, disjoint counting processes never jump together, so are independent by above, and $\phi := \sum_{t \geq 0} \delta_{(e(t),t)}$ is a Poisson measure with intensity $\mu = \nu \times dt$.

Lévy Processes and the Lévy-Khintchine Formula.

We can now sketch the close link between the general Lévy process on the one hand and the general infinitely-divisible law given by the Lévy-Khintchine formula (L-K) on the other.

First, if $X = (X_t)$ is Lévy, the law of each X_1 is infinitely divisible, so given by

$$E \exp\{iuX_1\} = \exp\{-\Psi(u)\} \qquad (u \in \mathbf{R})$$

with Ψ a Lévy exponent as in $(L - K)$. Similarly,

$$E \exp\{iuX_t\} = \exp\{-t\Psi(u)\} \qquad (u \in \mathbf{R}),$$

for rational t at first and general t by approximation and càdlàg paths. Then Ψ is called the *Lévy exponent*, or *characteristic exponent*, of the Lévy process

X. Conversely, given a Lévy exponent $\Psi(u)$ as in $(L-K)$, III.7 L24, construct a Brownian motion as in III.5 L20-22, and an independent Poisson point process $\Delta = (\Delta_t : t \geq 0)$ with characteristic measure μ, the Lévy measure in $(L-K)$. Then $X_1(t) := at + \sigma B_t$ has CF

$$E\exp\{iuX_1(t)\} = \exp\{-t\Psi_1(t)\} = \exp\left\{-t(iau + \frac{1}{2}\sigma^2 u^2)\right\},$$

giving the non-integral terms in $(L-K)$. For the 'large' jumps of Δ, write

$$\Delta_t^{(2)} := \begin{cases} \Delta_t & \text{if} \quad |\Delta_t| \geq 1, \\ 0 & \text{else.} \end{cases}$$

Then $\Delta^{(2)}$ is a Poisson point process with characteristic measure $\mu^{(2)}(dx) := I(|x| \geq 1)\mu(dx)$. Since $\int \min(1, |x|^2)\mu(dx) < \infty$, $\mu^{(2)}$ has finite mass, so $\Delta^{(2)}$, a $Ppp(\mu^{(2)})$, is discrete and its counting process

$$X_t^{(2)} := \sum_{s \leq t} \Delta_s^{(2)} \qquad (t \geq 0)$$

is compound Poisson, with Lévy exponent

$$\Psi^{(2)}(u) = \int (1 - e^{iux}) I(|x| \geq 1)\mu(dx) = \int (1 - e^{iux})\mu^{(2)}(dx).$$

There remain the 'small jumps',

$$\Delta_t^{(3)} := \begin{cases} \Delta_t & \text{if} \quad |\Delta_t| < 1, \\ 0 & \text{else.} \end{cases}$$

a $Ppp(\mu^{(3)})$, where $\mu^{(3)}(dx) = I(|x| < 1)\mu(dx)$, and independent of $\Delta^{(2)}$ because $\Delta^{(2)}$, $\Delta^{(3)}$ are Poisson point processes that never jump together. For each $\epsilon > 0$, the 'compensated sum of jumps'

$$X_t^{(\epsilon,3)} := \sum_{s \leq t} I(\epsilon < |\Delta_s| < 1)\Delta_s - t\int xI(\epsilon < |x| < 1)\mu(dx) \qquad (t \geq 0)$$

is a Lévy process with Lévy exponent

$$\Psi^{(\epsilon,3)}(u) = \int (1 - e^{iux} + iux) I(\epsilon < |x| < 1)\mu(dx).$$

Use of a suitable maximal inequality allows passage to the limit $\epsilon \downarrow 0$ (going from finite to possibly countably infinite sums of jumps): $X_t^{(\epsilon,3)} \to X_t^{(3)}$, a Lévy process with Lévy exponent

$$\Psi^{(3)}(u) = \int (1 - e^{iux} + iux)I(|x| < 1)\mu(dx),$$

independent of $X^{(2)}$ and with càdlàg paths. Combining:

Theorem. For $a \in \mathbf{R}, \sigma \geq 0, \int \min(1, |x|^2\mu(dx) < \infty$ and

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI(|x| < 1)\mu(dx),$$

the construction above yields a Lévy process

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

with Lévy exponent $\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$. Here the $X^{(i)}$ are independent Lévy processes, with Lévy exponents $\Psi^{(i)}$; $X^{(1)}$ is Gaussian, $X^{(2)}$ is a compound Poisson process with jumps of modulus ≥ 1; $X^{(3)}$ is a compensated sum of jumps of modulus < 1. The jump process $\Delta X = (\Delta X_t : t \geq 0)$ is a $Ppp(\mu)$, and similarly $\Delta X^{(i)}$ is a $Ppp(\mu^{(i)})$ for $i = 2, 3$.

Subordinators. We resort to complex numbers in the CF $\phi(u) = E(e^{iuX})$ because this always exists – for all real u – unlike the ostensibly simpler moment-generating function (MGF) $M(u) := E(e^{uX})$, which may well diverge for some real u. However, if the random variable X is *non-negative*, then for $s \geq 0$ the *Laplace-Stieltjes transform* (LST)

$$\psi(s) := E[e^{-sX}] \leq E(1) = 1$$

always exists. For $X \geq 0$ we have both the CF and the LST to hand, but the LST is usually simpler to handle. We can pass from CF to LST formally by taking $u = is$, and this can be justified by analytic continuation.

Some Lévy processes X have increasing (i.e. non-decreasing) sample paths; these are called *subordinators*. From the construction above, subordinators can have no negative jumps, so μ has support in $(0, \infty)$ and no mass on $(-\infty, 0)$. Because increasing functions have FV, one must have paths of (locally) finite variation, the condition for which can be shown to be

$$\int \min(1, |x|)\mu(dx) < \infty.$$

Thus the Lévy exponent must be of the form

$$\Psi(u) = -idu + \int_0^\infty (1 - e^{iux})\mu(dx),$$

with $d \geq 0$. It is more convenient to use the Laplace exponent $\Phi(s) = \Psi(is)$:

$$E\left(\exp\{-sX_t\}\right) = \exp\{-t\Phi(s)\} \qquad (s \geq 0), \qquad \Phi(s) = ds + \int_0^\infty (1 - e^{-sx})\mu(dx).$$

Example. The Stable Subordinator. Here $d = 0, \Phi(s) = s^\alpha,\ (0 < \alpha < 1)$,

$$\mu(dx) = dx/(\Gamma(1-\alpha)x^{\alpha-1}).$$

The special case $\alpha = 1/2$ is particularly important: this arises as the first-passage time of Brownian motion over positive levels, and gives rise to the Lévy density of Problems 9.

Classification.
IV (Infinite Variation). The sample paths have infinite variation on finite time-intervals, a.s. This occurs iff

$$\sigma > 0 \quad \text{or} \quad \int \min(1, |x|)\mu(dx) = \infty.$$

So take $\sigma = 0$ below.
FV (Finite Variation, on finite time-intervals, a.s.).

$$\int \min(1, |x|)\mu(dx) < \infty.$$

IA (Infinite Activity). Here there are infinitely many jumps in finite time-intervals, a.s.: μ has infinite mass, equivalently $\int_{-1}^1 \mu(dx) = \infty$:

$$\mu(\mathbf{R}) = \infty.$$

FA (Finite Activity). Here there are only finitely many jumps in finite time, a.s., and we are in the compound Poisson case:

$$\mu(\mathbf{R}) < \infty.$$