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Lecture 26. 6.12.2010

IV. Stochastic integration; Itô calculus.

1. Stochastic Integration

Stochastic integration was introduced by K. Itô in 1944, hence its name Itô calculus. It gives a meaning to

$$\int_0^t X dY = \int_0^t X(s,\omega) dY(s,\omega),$$

for suitable stochastic processes X and Y, the integrand and the integrator. We shall confine our attention here mainly to the basic case with integrator Brownian motion: $Y = W$. Much greater generality is possible; see e.g. [P] for details.

The first thing to note is that stochastic integrals with respect to Brownian motion, if they exist, must be quite different from the measure-theoretic integral of Ch. I. For, the Lebesgue-Stieltjes integrals described there have as integrators the difference of two monotone (increasing) functions, which are locally of finite variation. But we know from Ch. III that Brownian motion is of infinite (unbounded) variation on every interval. So Lebesgue-Stieltjes and Itô integrals must be fundamentally different.

In view of the above, it is quite surprising that Itô integrals can be defined at all. But if we take for granted Itô's fundamental insight that they can be, it is obvious how to begin and clear enough how to proceed. We begin with the simplest possible integrands X, and extend successively much as we extended the measure-theoretic integral of Ch. I.

Indicators. If $X(t,\omega) = I_{[a,b]}(t)$, there is exactly one plausible way to define $\int X dW$:

$$\int_0^t X(s,\omega) dW(s,\omega) := \begin{cases} 0 & \text{if } t \le a, \\ W(t) - W(a) & \text{if } a \le t \le b, \\ W(b) - W(a) & \text{if } t \ge b. \end{cases}$$

Simple Functions. Extend by linearity: if X is a linear combination of indicators, $X = \sum_{i=1}^n c_i I_{[a_i,b_i]}$, we should define

$$\int_0^t X dW := \sum_{i=1}^n c_i \int_0^t I_{[a_i,b_i]} dW.$$

Already one wonders how to extend this from constants c_i to suitable random variables, and one seeks to simplify the obvious but clumsy three-line expressions above.

We begin again, this time calling a stochastic process X *simple* if there is a partition $0 = t_0 < t_1 < \ldots < t_n = T < \infty$ and uniformly bounded $\mathcal{F}_{t_n}$-measurable random variables ξ_k ($|\xi_k| \le C$ for all $k = 0, \ldots, n$ and ω, for some C) and if $X(t, \omega)$ can be written in the form

$$X(t,\omega) = \xi_0(\omega) I_{\{0\}}(t) + \sum_{i=0}^{n} \xi_i(\omega) I_{(t_i, t_{i+1}]}(t) \qquad (0 \le t \le T, \omega \in \Omega).$$

Then if $t_k \le t < t_{k+1}$,

$$\begin{aligned} I_t(X) := \textstyle\int_0^t X dW &= \textstyle\sum_{i=0}^{k-1} \xi_i W(t_{i+1}) - W(t_i)) + \xi_k (W(t) - W(t_k)) \\ &= \textstyle\sum_{i=0}^{n} \xi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)). \end{aligned}$$

Note that by definition $I_0(X) = 0$ a.s. We collect some properties of the stochastic integral defined so far:

Lemma. (i) $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.
(ii) $E(I_t(X)|\mathcal{F}_s) = I_s(X) \qquad a.s. \qquad (0 \le s < t < \infty)$, hence $I_t(X)$ is a *continuous martingale*.

Proof. (i) follows from the fact that linear combinations of simple functions are simple.
(ii) There are two cases to consider.
(a) Both s and t belong to the same interval $[t_k, t_{k+1})$. Then

$$I_t(X) = I_s(X) + \xi_k (W(t) - W(s)).$$

But ξ_k is $\mathcal{F}_{t_k}$-measurable, so $\mathcal{F}_s$-measurable ($t_k \le s$), so independent of $W(t) - W(s)$ (independent increments property of W). So

$$E(I_t(X)|\mathcal{F}_s) = I_s(X) + \xi_k E(W(t) - W(s)|\mathcal{F}_s) = I_s(X).$$

(b) $s < t$ belongs to a different interval from t: $s \in [t_m, t_{m+1})$ for some $m < k$. Then

$$I_t(X) = I_s(X) + \xi_m (W(t_{m+1}) - W(s)) + \sum_{i=m+1}^{k-1} \xi_i (W(t_{i+1}) - W(t_i)) + \xi_k (W(t) - W(t_k))$$

(if $k = m + 1$, the sum on the right is empty, and does not appear). Take $E(.|\mathcal{F}_s)$ on the right. The first term gives $I_s(X)$. The second gives $\xi_m E[(W(t_{m+1}) - W(s))|\mathcal{F}_s] = \xi_m \cdot 0 = 0$, as ξ_m is $\mathcal{F}_s$-measurable, and similarly so do the third and fourth, completing the proof. //

Note. The stochastic integral for simple integrands is essentially a martingale transform.

We pause to note a property of square-integrable martingales which we shall need below. Call $M(t) - M(s)$ the increment of M over $(s, t]$. Then for a martingale M, the product of the increments over disjoint intervals has zero mean. For, if $s < t \le u < v$,

$$E[(M(v) - M(u))(M(t) - M(s))] = E[E((M(v) - M(u))(M(t) - M(s))|\mathcal{F}_u)]$$

$$= E[(M(t) - M(s))E((M(v) - M(u))|\mathcal{F}_u)],$$

taking out what is known (as $s, t \le u$). The inner expectation is zero by the martingale property, so the left-hand side is zero, as required.

We now can add further properties of the stochastic integral for simple functions X.

Lemma. (i) We have the Itô isometry

$$E[(I_t(X))^2], \text{ or } E[(\int_0^t X dW)^2], = E(\int_0^t X(s)^2 ds).$$

(ii) $E((I_t(X) - I_s(X))^2|\mathcal{F}_s) = E(\int_s^t X(u)^2 du)$ a.s.

Proof. We only show (i); the proof of (ii) is similar. The left-hand side in (i) above is $E(I_t(X) \cdot I_t(X))$, i.e.

$$E([\sum_{i=0}^{k-1} \xi_i (W(t_{i+1}) - W(t_i)) + \xi_k (W(t) - W(t_k))]^2).$$

Expanding out the square, the cross-terms have expectation zero by above, leaving

$$E(\sum_{i=0}^{k-1} \xi_i^2 (W(t_{i+1}) - W(t_i))^2 + \xi_k^2 (W(t) - W(t_k))^2).$$

Since ξ_i is $\mathcal{F}_{t_i}$-measurable, each ξ_i^2-term is independent of the squared Brownian increment term following it, which has expectation $var(W(t_{i+1})-W(t_i)) = t_{i+1} - t_i$. So we obtain

$$\sum_{i=0}^{k-1} E(\xi_i^2)(t_{i+1} - t_i) + E(\xi_k^2)(t - t_k).$$

This is (using Fubini's theorem) $\int_0^t E(X(u)^2)du = E(\int_0^t X(u)^2 du)$, as required. //

The Itô isometry above suggests that $\int_0^t XdW$ should be defined only for processes with

$$\int_0^t E(X(u)^2)du < \infty \qquad \text{for all } t. \qquad (*)$$

We then can transfer convergence on a suitable L^2-space of stochastic processes to a suitable L^2-space of martingales. This gives us an L^2-theory of stochastic integration, for which Hilbert-space methods are available.

Approximation. By analogy with the integral of Ch. I, we seek a class of integrands suitably approximable by simple integrands. It turns out that:
(i) The suitable class of integrands is the class of $(\mathcal{B}([0,\infty)) \times \mathcal{F})$-measurable, $(\mathcal{F}_t)$- adapted processes X with $\int_0^t E(X(u)^2)du < \infty$ for all $t > 0$.
(ii) Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t XdW$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dW$.
(iii) The properties from both lemmas above remain true for the stochastic integral $\int_0^t XdW$ defined by (i) and (ii).
We must omit detailed proofs of these assertions here. The key technical ingredients needed are Hilbert-space methods in spaces defined by integrals related to the quadratic variation of the integrator (which is just t in our Brownian motion setting here) and the Kunita-Watanabe inequalities ([P], 61).

Without (∗), the stochastic integral need not yield a mg, but only a *local martingale*. This is a process M such that there exists a sequence of stopping times $T_n \uparrow +\infty$ such that each of the stopped and shifted processes $M^{T_n} - M_0$ is a (true) martingale. Local mgs are much more general than (true) mgs. They are used to define *semi-martingales* – sums of a local mg and a FV process; these are the most general *stochastic integrators* [L27, L30].