spl27.tex

Lecture 27. 10.12.2010

Example. We calculate $\int W(u)dW(u)$. We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} W(0) = 0 & \text{if} \quad 0 \le u \le t/n, \\ W(t/n) & \text{if} \quad t/n < u \le 2t/n, \\ \vdots & \vdots \\ W((n-1)t/n) & \text{if} \quad (n-1)t/n < u \le t. \end{cases}$$

By definition,

$$\int_0^t W(u)dW(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} W(kt/n)(W((k+1)t/n) - W(kt/n)).$$

Replacing $W(kt/n)$ by $\frac{1}{2}(W((k+1)t/n) + W(kt/n)) - \frac{1}{2}(W((k+1)t/n) - W(kt/n))$, the RHS is

$$\sum \frac{1}{2}(W((k+1)t/n) + W(kt/n)).(W((k+1)t/n) - W(kt/n))$$

$$-\sum \frac{1}{2}(W((k+1)t/n) - W(kt/n)).(W((k+1)t/n) - W(kt/n)).$$

The first sum is $\sum \frac{1}{2}(W((k+1)t/n)^2 - W(kt/n)^2)$, which telescopes (as a sum of differences) to $\frac{1}{2}W(t)^2$ ($W(0) = 0$). The second sum is $\frac{1}{2}\sum(W(k+1)t/n) - W(kt/n))^2$, an approximation to the quadratic variation of W on $[0, t]$, which tends to $\frac{1}{2}t$ by Lévy's theorem on the QV. Combining,

$$\int_0^t W(u)dW(u) = \frac{1}{2}W(t)^2 - \frac{1}{2}t.$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of W.

One can construct a closely analogous theory for stochastic integrals with the Brownian integrator W above replaced by a square-integrable martingale integrator M. The properties above hold, with the Lemma (i) replaced by

$$E[(\int_0^t X(u)dM(u))^2] = E[\int_0^t X(u)^2 d\langle M\rangle(u)].$$

The natural class of integrands X to use here is the class of predictable processes (a slight extension of left-continuity of sample paths).

Quadratic Variation, Quadratic Covariation. We shall need to extend quadratic variation and quadratic covariation to stochastic integrals. The quadratic variation of $I_t(X) = \int_0^t X(u)dW(u)$ is $\int_0^t X(u)^2 du$. This is proved in the same way as the case $X \equiv 1$, that W has quadratic variation process t. More generally, if $Z(t) = \int_0^t X(u)dM(u)$ for a continuous martingale integrator M, then $\langle Z\rangle(t) = \int_0^t X^2(u)d\langle M\rangle(u)$. Similarly (or by polarization), if $Z_i(t) = \int_0^t X_i(u)dM_i(u)$ $(i = 1, 2)$, $\langle Z_1, Z_2\rangle(t) = \int_0^t X_1(u)X_2(u)d\langle M_1, M_2\rangle(u)$.

Semi-martingales.

It turns out that semi-martingales give the natural class of stochastic integrators: one can define the stochastic integral

$$\int_0^t H(u)dX(u) = \int_0^t H(u)dM(u) + \int_0^t H(u)dA(u)$$

for predictable integrands H (as above), and for semi-martingale integrators X – but for no larger class of integrators, if one is to preserve reasonable convergence and approximation properties for the operation of stochastic integration. For details, see e.g. [P].

With integrands as general as above, stochastic integrals are no longer martingales in general, but only *local martingales* (see e.g. [P]: martingales on each $[0, T_n]$, for some sequence of stopping times $T_n \uparrow \infty$). For our purposes, one loses little by thinking of bounded integrands (recall that we usually have a finite time horizon T, the expiry time of an option, and that bounded processes are locally integrable, but not integrable in general).

2. Itô's Lemma

Suppose that b is adapted and locally integrable (so $\int_0^t b(s)ds$ is defined as an ordinary integral, as in Ch. I), and σ is adapted and measurable with $\int_0^t E(\sigma(u)^2)du < \infty$ for all t (so $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral, as above). Then

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$ (which is often called an Itô process). It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \qquad X(0) = x_0. \tag{$*$}$$

Now suppose $f : \mathbf{R} \to \mathbf{R}$ is of class C^2. The question arises of giving a meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$, and finding it. Given a partition $\mathcal{P}$ of $[0, t]$, i.e. $0 = t_0 < t_1 < \ldots < t_n = t$, we can use Taylor's formula to obtain

$$f(X(t)) - f(X(0)) = \sum_{k=0}^{n-1} f(X(t_{k+1})) - f(X(t_k))$$

$$= \sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) + \frac{1}{2}\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2$$

with $0 < \theta_k < 1$. We know that $\sum(\Delta X(t_k))^2 \to \langle X \rangle(t)$ in probability (so, taking a subsequence, with probability one), and a little more effort gives

$$\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \to \int_0^t f''(X(u))d\langle X \rangle(u).$$

The first sum is easily recognized as an approximating sequence of a stochastic integral (compare the example above), giving

$$\sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \to \int_0^t f'(X(u))dX(u) :$$

Theorem (Basic Itô Formula). If X has stochastic differential given by $(*)$ and $f \in C^2$, then $f(X)$ has stochastic differential

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d\langle X \rangle(t),$$

or writing out the integrals,

$$f(X(t)) = f(x_0) + \int_0^t f'(X(u))dX(u) + \frac{1}{2}\int_0^t f''(X(u))d\langle X \rangle(u).$$

More generally, suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. By the Taylor expansion of a smooth function of several variables we get for t close to t_0 (we use subscripts to denote partial derivatives: $f_t := \partial f/\partial t$, $f_{tx} := \partial^2 f/\partial t \partial x$):

$$\begin{aligned} f(t, X(t)) &= f(t_0, X(t_0)) \\ &\quad +(t-t_0) f_t(t_0, X(t_0)) + (X(t) - X(t_0)) f_x(t_0, X(t_0)) \\ &\quad +\tfrac{1}{2}(t-t_0)^2 f_{tt}(t_0, X(t_0)) + \tfrac{1}{2}(X(t) - X(t_0))^2 f_{xx}(t_0, X(t_0)) \\ &\quad +(t-t_0)(X(t) - X(t_0)) f_{tx}(t_0, X(t_0)) + \ldots, \end{aligned}$$

which may be written symbolically as

$$df = f_t dt + f_x dX + \frac{1}{2} f_{tt} (dt)^2 + f_{tx} dt dX + \frac{1}{2} f_{xx} (dX)^2 + \ldots.$$

In this, we substitute $dX(t) = b(t)dt + \sigma(t)dW(t)$ from above, to obtain

$$\begin{aligned} df &= f_t dt + f_x (bdt + \sigma dW) \\ &\quad +\tfrac{1}{2} f_{tt} (dt)^2 + f_{tx} dt (bdt + \sigma dW) + \tfrac{1}{2} f_{xx} (bdt + \sigma dW)^2 + \ldots \end{aligned}$$

Now using the formal multiplication rules $dt \cdot dt = 0, dt \cdot dW = 0, dW \cdot dW = dt$ (which are just shorthand for the corresponding properties of the quadratic variations), we expand

$$(bdt + \sigma dW)^2 = \sigma^2 dt + 2b\sigma dt dW + b^2 (dt)^2 = \sigma^2 dt + \text{higher-order terms}$$

to get finally

$$df = \left(f_t + b f_x + \frac{1}{2} \sigma^2 f_{xx} \right) dt + \sigma f_x dW + \text{higher-order terms}.$$

As above, the higher-order terms are irrelevant, and summarizing, we obtain *Itô's lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem (Itô's Lemma). If $X(t)$ has stochastic differential given by $(*)$, then $f = f(t, X(t))$ has stochastic differential

$$df = \left(f_t + b f_x + \frac{1}{2} \sigma^2 f_{xx} \right) dt + \sigma f_x dW.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f = f_0 + \int_0^t (f_t + b f_x + \frac{1}{2} \sigma^2 f_{xx}) dt + \int_0^t \sigma f_x dW. \qquad //$$