spl28.tex
Lecture 28. 13.12.2010

We will make good use of:

Corollary. $E(f(t,X(t))) = f_0 + \int_0^t E(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx})dt$.

Proof. $\int_0^t \sigma f_2 dW$ is a stochastic integral, so a (local) martingale, so its expectation is constant (= 0, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's lemma really comes into its own in the more general setting of semi-martingales (of which X above is an important example). It says there that if X is a semi-martingale and f is a smooth function as above, then $f(t,X(t))$ is also a semi-martingale. The ordinary differential dt gives rise to the finite-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.

Itô Lemma in Higher Dimensions. If $f(t,x_1,\ldots,x_d)$ is C^1 in its zeroth (time) argument t and C^2 in its remaining d space arguments x_i, and $M = (M_1,\ldots,M_d)$ is a continuous vector martingale, then (writing f_i, f_{ij} for the first partial derivatives of f with respect to its ith argument and the second partial derivatives with respect to the ith and jth arguments) $f(t,M(t))$ has stochastic differential

$$df(t,M(t)) = f_0(t,M(t))dt + \sum_{i=1}^d f_i(t,M(t))dM_i(t) + \frac{1}{2}\sum_{i,j=1}^d f_{ij}(t,M(t))d\langle M_i,M_j\rangle(t).$$

Application. The case $f(x) = x^2$ gives

$$W(t)^2 = W(0)^2 + \int_0^t 2W(u)dW(u) + \frac{1}{2}\int_0^t 2du,$$

which after rearranging is just our earlier example.

3. Geometric Brownian Motion

Now that we have both Brownian motion W and Itô's Lemma to hand, we can introduce the most important stochastic process for us, a relative of

Brownian motion – *geometric* (or *exponential*, or *economic*) Brownian motion.

To model the stock-price evolution, we use the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \qquad S(0) > 0,$$

due to Itô in 1944. (Interpretation [see handout]: the return dS/S over a short time-interval is the sum of the deterministic term μdt and the random term σdW.) This corrects Bachelier's earlier attempt of 1900 (he did not have the factor $S(t)$ on the right - missing the interpretation in terms of returns, and leading to negative stock prices!) Incidentally, Bachelier's work served as Itô's motivation in introducing Itô calculus. The mathematical importance of Itô's work was recognised early, and led on to the work of Doob in 1953 [D], Meyer (1960s on) and many others. The economic importance of geometric Brownian motion was recognized by Paul A. Samuelson in his work from 1965 on, for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton, in work for which he was similarly honoured in 1997.

The differential equation above has the unique solution

$$S(t) = S(0) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma dW(t)\right\}.$$

For, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right) f, \qquad f_x = \sigma f, \qquad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \qquad (dx)^2 = dt.$$

Thus Itô's lemma gives

$$\begin{aligned} df(t, W(t)) &= f_t dt + f_x dW(t) + \tfrac{1}{2} f_{xx} (dW(t))^2 \\ &= f\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dW(t) + \tfrac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW(t)), \end{aligned}$$

so $f(t, W(t))$ is a solution of the stochastic differential equation, and the initial condition $f(0, W(0)) = S(0)$ as $W(0) = 0$, giving existence.

For uniqueness, we need the *stochastic* (or Doléans, or Doléans-Dade) *exponential* (below), giving $Y = \mathcal{E}(X) = \exp\{X - \frac{1}{2}\langle X \rangle\}$ (with X a continuous semi-martingale) as the unique solution to the stochastic differential equation

$$dY(t) = Y(t-)dX(t), \qquad Y(0) = 1.$$

(Incidentally, this is one of the few cases where a stochastic differential equation can be solved explicitly. Usually we must be content with an existence and uniqueness statement, and a numerical algorithm for calculating the solution.) Thus $S(t)$ above is the stochastic exponential of $\mu t + \sigma W(t)$, Brownian motion with mean (or drift) μ and variance (or volatility) σ^2. In particular,

$$\log S(t) = \log S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)$$

has a normal distribution. Thus $S(t)$ itself has a *lognormal* distribution. This geometric Brownian motion model, and the log-normal distribution that it entails, are the basis for the Black-Scholes model for stock-price dynamics in continuous time.

4. Stochastic Calculus for Black-Scholes Models; Girsanov's theorem

In this section we collect the main tools for the analysis of financial markets with uncertainty modelled by Brownian motions.

Consider first independent $N(0,1)$ random variables $Z_1, \ldots, Z_n$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\gamma = (\gamma_1, \ldots, \gamma_n)$, consider a new probability measure Q on $(\Omega, \mathcal{F})$ defined by

$$Q(d\omega) = \exp\left\{\sum_{i=1}^{n} \gamma_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2\right\} P(d\omega).$$

As $\exp\{.\} > 0$ and integrates to 1, as $\int \exp\{\gamma_i Z_i\} dP = \exp\{\frac{1}{2}\gamma_i^2\}$, this is a probability measure. It is also equivalent to P (has the same null sets), again as the exponential term is positive. Also

$$Q(Z_i \in dz_i, \quad i = 1, \ldots, n) = \exp\left\{\sum_{i=1}^{n} \gamma_i Z_i - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2\right\} P(Z_i \in dz_i, \ i = 1, \ldots, n)$$

$$= (2\pi)^{-n/2} \exp\left\{\sum_{i=1}^{n} \gamma_i z_i - \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2 - \frac{1}{2}\sum_{i=1}^{n} z_i^2\right\} \prod_{i=1}^{n} dz_i$$

$$= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(z_i - \gamma_i)^2\right\} dz_1 \dots dz_n.$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\gamma_i, 1)$ under Q. Thus the effect of the *change of measure* $P \to Q$, from the original measure P to the *equivalent* measure Q, is to *change the mean*, from $0 = (0, \dots, 0)$ to $\gamma = (\gamma_1, \dots, \gamma_n)$.

This result extends to infinitely many dimensions. Let $W = (W_1, \dots W_d)$ be a d-dimensional Brownian motion defined on a stochastic basis with the filtration satisfying the usual conditions. Let $(\gamma(t) : 0 \le t \le T)$ be a measurable, adapted d-dimensional process with $\int_0^T \gamma_i(t)^2 dt < \infty$ a.s., $i = 1, \dots, d$, and define the process $(L(t) : 0 \le t \le T)$ by

$$L(t) = \exp\left\{-\int_0^t \gamma(s)' dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)^2\| ds\right\}.$$

Then L is continuous, and, being the stochastic exponential of $-\int_0^t \gamma(s)' dW(s)$, is a local martingale. Given sufficient integrability on the process γ, L will in fact be a (continuous) martingale. For this, *Novikov's condition* suffices:

$$E\left(\exp\left\{\frac{1}{2}\int_0^T \|\gamma(s)^2\| ds\right\}\right) < \infty.$$

We are now in the position to state a version of Girsanov's theorem, which is one of the main tools in studying continuous-time financial market models.

Theorem (Girsanov). Let γ be as above and satisfy Novikov's condition; let L be the corresponding continuous martingale. Define the processes $\tilde{W}_i$, $i = 1, \dots, d$ by

$$\tilde{W}_i(t) := W_i(t) + \int_0^t \gamma_i(s) ds, \qquad (0 \le t \le T), \qquad i = 1, \dots, d.$$

Then under the equivalent probability measure Q defined on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym derivative

$$\frac{dQ}{dP} = L(T),$$

the process $\tilde{W} = (\tilde{W}_1, \dots, \tilde{W}_d)$ is d-dimensional Brownian motion.