spl29.tex
Lecture 29. 13.12.2010

In particular, for $\gamma(t)$ constant ($=\gamma$), change of measure by introducing the Radon-Nikodym derivative $\exp\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\}$ corresponds to a change of drift from c to $c-\gamma$. If $(\mathcal{F}_t)$ is the Brownian filtration (basically $\mathcal{F}_t = \sigma(W(s), 0 \le s \le t)$ slightly enlarged to satisfy the usual conditions) any pair of equivalent probability measures $Q \sim P$ on $\mathcal{F} = \mathcal{F}_\mathcal{T}$ is a Girsanov pair, i.e.

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = L(t)$$

with L defined as above.
Note. The main application of the Girsanov theorem in mathematical finance is the change of measure in the Black-Scholes model of a financial market to obtain the risk-neutral martingale measure, under which discounted asset prices give prices of derivatives (options etc.). The relevant mathematics needed includes the following.

Theorem (Brownian Martingale Representation Theorem). Let $M = (M(t))_{t\ge 0}$ be a RCLL local martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then

$$M(t) = M(0) + \int_0^t H(s)dW(s),\ t \ge 0$$

with $H = (H(t))_{t\ge 0}$ a progressively measurable process such that $\int_0^t H(s)^2 ds < \infty,\ t \ge 0$ with probability one. That is, all Brownian local martingales may be represented as stochastic integrals with respect to Brownian motion (and as such are continuous).

As mentioned above, the economic relevance of the representation theorem is that it shows that the Black-Scholes model is complete – that is, that every contingent claim (modelled as an appropriate random variable) can be replicated by a dynamic trading strategy. Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of Brownian motion are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

The next result, which is an example for the rich interplay between probability theory and analysis, links stochastic differential equations (SDEs) with partial differential equations (PDEs). Such links between probability and stochastic processes on the one hand and analysis and partial differential equations on the other are very important, and have been extensively studied. Suppose we consider a stochastic differential equation,

$$dX(t) = \mu(t, X(t))ds + \sigma(t, X(t))dW(t) \qquad (t_0 \le t \le T), \qquad X(t_0) = x.$$

For suitably well-behaved functions μ, σ, this stochastic differential equation will have a unique solution $X = (X(t) : t_0 \le t \le T)$. Taking existence of a unique solution for granted for the moment, consider a smooth function $F(t, X(t))$ of it. By Itô's lemma,

$$dF = F_t dt + F_x dX + \frac{1}{2} F_{xx} d\langle X\rangle,$$

and as $d\langle X\rangle = \langle \mu dt + \sigma dW\rangle = \sigma^2 d\langle W\rangle = \sigma^2 dt$, this is

$$dF = F_t dt + F_x(\mu dt + \sigma dW) + \frac{1}{2}\sigma^2 F_{xx} dt = (F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx})dt + \sigma F_x dW.$$

Now suppose that F satisfies the partial differential equation

$$F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} = 0$$

with boundary condition,

$$F(T, x) = h(x).$$

Then the above expression for dF gives

$$dF = \sigma F_x dW,$$

which can be written in stochastic-integral rather than stochastic-differential form as

$$F(s, X(s)) = F(t_0, X(t_0)) + \int_{t_0}^{s} \sigma(u, X(u)) F_x(u, X(u)) dW(u).$$

Under suitable conditions, the stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Then

$$F(t_0, x) = E\left(F(s, X(s)) \middle| X(t_0) = x \right).$$

For simplicity, we restrict to the time-homogeneous case: $\mu(t,x) = \mu(x)$ and $\sigma(t,x) = \sigma(x)$, and assume μ and σ Lipschitz, and $h \in C_0^2$ (h twice continuously differentiable, with compact support). Then the stochastic integral is a martingale, and replacing t_0, s by t, T we get the stochastic representation $F(t,x) = E\left(F(X(T))|\, X(t) = x\right)$ for the solution F. Conversely, any solution F which is in $C^{1,2}$ (has continuous derivatives of order one in t and two in x) and is bounded on compact t-sets arises in this way. This gives:

Theorem (Feynman-Kac Formula. For $\mu(x), \sigma(x)$ Lipschitz, the solution $F = F(t,x)$ to the partial differential equation

$$F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} = 0$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = E\left[h(X(T))|\, X(t) = x\right],$$

where X satisfies the stochastic differential equation

$$dX(s) = \mu(X(s))ds + \sigma(X(s))dW(s) \qquad (t \le s \le T)$$

with initial condition $X(t) = x$.

The Feynman-Kac formula gives a stochastic representation to solutions of partial differential equations (e.g., the Black-Scholes PDE).

Application. One classical application of the Feynman-Kac formula is to Kac's proof of Lévy's arc-sine law for Brownian motion. Let τ_t be the amount of time in $[0,t]$ for which Brownian motion takes positive values. Then the proportion τ_t/t has the *arc-sine* law - the law on $[0,1]$ with density $1/(\pi\sqrt{x(1-x)})$ $(x \in [0,1)$.

5. Stochastic Differential Equations

Perhaps the most basic general existence theorem for SDEs is Picard's theorem, for an ordinary differential equation (non-linear, in general)

$$dx(t) = b(t, x(t))dt, \qquad x(0) = x_0,$$

or to use its alternative and equivalent expression as an integral equation,

$$x(t) = x_0 + \int_0^t b(s, x(s))ds.$$

If one assumes the *Lipschitz condition*

$$|b(t,x) - b(t,y)| \leq K|x - y|$$

for some constant K and all $t \in [0, T]$ for some $T > 0$, and boundedness of b on compact sets, one can construct a unique solution x by the *Picard iteration*

$$x^{(0)}(t) := x_0, \qquad x^{(n+1)}(t) := x_0 + \int_0^t b(s, x^{(n)}(s))ds.$$

See any textbook on analysis or differential equations. (The result may also be obtained as an application of Banach's contraction-mapping principle in functional analysis.)

Naturally, stochastic calculus and stochastic differential equations contain all the complications of their non-stochastic counterparts, and more besides. Thus by analogy with PDEs alone, we must expect study of SDEs to be complicated by the presence of more than one concept of a solution. The first solution concept that comes to mind is that obtained by sticking to the non-stochastic theory, and working pathwise: take each sample path of a stochastic process as a function, and work with that. This gives the concept of a *strong* solution of a stochastic differential equation. Here we are given the probabilistic set-up – the filtered probability space in which our SDE arises – and work within it. The most basic results, like their non-stochastic counterparts, assume regularity of coefficients (e.g., Lipschitz conditions), and construct a unique solution by a stochastic version of Picard iteration. Consider the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \qquad X(0) = \xi,$$

where $b(t,x)$ is a d-vector of drifts, $\sigma(t,x)$ is a $d \times r$ dispersion matrix, $W(t)$ is an r-dimensional Brownian motion, ξ is a square-integrable random d-vector independent of W, and we work on a filtered probability space satisfying the usual conditions on which W and ξ are both defined. Suppose that the coefficients b, σ satisfy the following global Lipschitz and growth conditions:

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|,$$

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq K^2(1 + \|x\|^2),$$

for all $t \geq 0,\ x, y \in \mathbf{R}^d$, for some constant $K > 0$.

Theorem. Under the above Lipschitz and growth conditions,
(i) the Picard iteration $X^{(0)}(t) := \xi$,

$$X^{(n+1)}(t) := \xi + \int_0^t b(s, X^{(n)}(s))ds + \int_0^t \sigma(s, X^{(n)}(s))dW(s)$$

converges, to $X(t)$ say;
(ii) $X(t)$ is the unique strong solution to the stochastic differential equation

$$X(0) = \xi, \qquad X(t) = \xi + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s);$$

(iii) $X(t)$ is square-integrable, and for each $T > 0$ there exists a constant C, depending only on K and T, such that $X(t)$ satisfies the growth condition

$$E\left(\|X(t)\|^2\right) \leq C\left(1 + E\left(\|\xi\|^2\right)\right)e^{Ct} \qquad (0 \leq t \leq T).$$

Unfortunately, it turns out that not all SDEs have strong solutions. However, in many cases one can nevertheless solve them, by setting up a filtered probability space for oneself, setting up an SDE of the required form on it, and solving the SDE there. The resulting solution concept is that of a *weak solution*. Naturally, weak solutions are distributional, rather than pathwise, in nature. However, it turns out that it is the weak solution concept that is often more appropriate for our purposes. This is particularly so in that we will often be concerned with convergence of a sequence of (discrete) financial models to a (continuous) limit. The relevant convergence concept here is that of *weak convergence*. In the continuous setting, the price dynamics are described by a stochastic differential equation, in a discrete setting by a stochastic difference equation. One seeks results in which weak solutions of the one converge weakly to weak solutions of the other.