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Lecture 3. 15.10.2010.

3. Measures.

Following Lebesgue (1902), our next task is to study the mathematics of length, area and volume (in Euclidean space of dimensions 1, 2, 3, understood). It turns out that to do this, we actually do much more.

Length/area/volume is non-negative, and defined on classes of sets (as in L2) – intervals/rectangles/cuboids in the first instance. The whole of Euclidean space has infinite length/area/volume, so we allow the value $+\infty$.

Defn. A *measure* μ on a σ-algebra $\mathcal{A}$ of subsets of a set Ω is a set-valued function $\mu : \mathcal{A} \to [0, \infty]$ such that
(i) $\mu(\emptyset) = 0$;
(ii) If A_n $(n = 1, 2, \ldots)$ are disjoint sets in $\mathcal{A}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n). \qquad (ca)$$

Defn. We then call $(\Omega, \mathcal{A})$ a *measurable space*, and $(\Omega, \mathcal{A}, \mu)$ a *measure space*. The sets $A \in \mathcal{A}$ (for which $\mu(A)$ is defined, as $\mu : \mathcal{A} \to [0, \infty]$) are called the *$\mu$-measurable sets*.

If $\mu(\Omega) < \infty$, we call the measure *finite*. If $\Omega = \bigcup_{n=1}^{\infty} A_n$ with $A_n \in \mathcal{S}$ increasing ($A_n \subset A_{n+1}$) and each $\mu(A_n) < \infty$, we call μ *σ-finite*.

If $\mu(\Omega) = 1$, we call μ a *probability measure*, or *probability* for short, and $(\Omega, \mathcal{A}, \mu)$ a *probability space*.

If μ satisfies (i) and (ii) but $\mathcal{A}$ is not a σ-algebra, we call μ a *pre-measure* on $\mathcal{A}$.

Note. If in (ii) we only take $\mathcal{A}$ as a field, rather than a σ-field, and correspondingly only ask for μ to add over disjoint unions of *two* sets –

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$$

for A_1, A_2 disjoint sets in $\mathcal{A}$, then using induction we can obtain

$$\mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n) \qquad (fa)$$

for any finite sequence of disjoint sets $A_n \in \mathcal{A}$. This property is called *finite additivity*, while (ii) above is called *countable additivity* (whence the names

(fa) and (ca)). Of course countable additivity is a stronger assumption than finite additivity (ca implies fa, but not conversely – note that one cannot use induction to get from N in (fa) to ∞ in (ca), as ∞ is not a positive integer), and of course stronger assumptions lead to stronger conclusions, or theorems. In Measure Theory, following Lebesgue and most later authors, we shall assume ca. But it is possible to build a theory instead on fa; for a recent comparison between the two, see
N. H. BINGHAM, Finite additivity versus countable additivity, *Electronic Journal for the History of Probability and Statistics* vol. **6** no. 1 (2010), 35p [www.jehps.net].

On the line, the length of an interval $(a, b]$ is $\mu((a, b]) := b - a$. How can we extend this formula to wider classes of sets – preferably, to as wide a class of sets as possible – while preserving the countable additivity property ca? The idea is to use small, easy-to-visualize classes of sets (such as the half-open intervals above) as generators $\mathcal{G}$ and build up systematically from there. It turns out that we need a class $\mathcal{G}$ to have certain properties for it to be suitable for such a purpose.
Defn. A class $\mathcal{S}$ is a *semi-ring* if
(i) $\emptyset \in \mathcal{S}$;
(ii) $\mathcal{S}$ is closed under intersections: if $A_1, A_2 \in \mathcal{S}$, then $A_1 \cap A_2 \in \mathcal{S}$;
(iii) if $A, B \in \mathcal{S}$, the set-theoretic difference $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$.

The motivating example here is the class of half-open intervals on the line, or its analogue in higher dimensions. On the line, the intersection of two half-open intervals is empty or again a half-open interval; the difference of two half-open intervals is empty, one or the disjoint union of two half-open intervals. Similarly in the plane: the difference of two half-open rectangles is a disjoint union of at most 8 ($= 3^2 - 1$) half-open rectangles (draw a picture). Similarly in 3-space: the difference of two half-open cuboids is a disjoint union of at most $26 = 3^3 - 1$ half-open cuboids, etc. (again, draw a picture – but pictures of 3-dimensional situations are harder to draw!) Similarly in d dimensions (where we draw pictures 'as if $d = 3'$).

The main result of Measure Theory is Carathéodory's Extension Theorem (Constantin Carathéodory (1873-1950), in his book of 1914). We quote this below; for details of the proof, see e.g. [S], Ch. 6.

Theorem (Carathéodory's Extension Theorem). Let $\mathcal{S}$ be a semi-ring of subsets of a set Ω, and $\mu : \mathcal{S} \to [0, \infty]$ a pre-measure, i.e. $\mu(\emptyset) = 0$, and if

A_n are disjoint sets in $\mathcal{S}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then μ can be extended to a measure (also called μ) on the σ-algebra $\sigma(\mathcal{S})$ generated by $\mathcal{S}$. This extension is unique if μ is σ-finite.

We include only a sketch proof, following Schilling's account [S].

For an arbitrary subset A of Ω, define

$$\mu^*(A) := \inf\{\sum \mu(S_n)\},$$

where the infimum extends over all coverings of A by sequences of sets $S_n \in \mathcal{S}$ (i.e. $A \subset \bigcup S_n$).

Step 1. (i) μ^* vanishes on the empty set:

$$\mu^*(\emptyset) = 0; \qquad (OM1)$$

(ii) μ^* is *monotone*: if $A \subset B$, then

$$\mu^*(A) \leq \mu^*(B); \qquad (OM2)$$

(iii) μ^* is σ-subadditive:

$$\mu^*(\bigcup A_n) \leq \sum \mu^*(A_n). \qquad (OM3)$$

We summarize $(OM1)$ – $(OM3)$ by saying that μ^* is an *outer measure* (whence (OM)).

Step 2. μ^* extends μ – i.e., agrees with μ on the sets in $\mathcal{S}$ on which they are both defined (but is itself defined on *all* sets in Ω).

Step 3. Write $\mathcal{A}^*$ for the class of μ^*-*measurable sets*, i.e. the sets $A \subset \Omega$ for which

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A)$$

for all sets B in Ω. Then $\mathcal{S} \subset \mathcal{A}^*$.

Step 4. $\mathcal{A}^*$ is a σ-algebra, and μ^* is a measure on $(\Omega, \mathcal{A}^*)$.

Step 5. μ^* is a measure on $\sigma(\mathcal{S})$ $(\subset \mathcal{A}^*)$.

Step 6. This extension is unique if μ is σ-finite.

Lebesgue measure.

On the line, the measure λ defined on the semi-ring of half-open intervals by

$$\lambda((a,b]) := b - a$$

is called *Lebesgue measure* on the line. It generalizes *length*. Similarly, its analogue for 2 dimensions,

$$\lambda((a_1, b_1] \times (a_2, b_2]) := (b_1 - a_1)(b_2 - a_2),$$

generalizes *area*, that for 3 dimensions *volume*, and that for d dimensions,

$$\lambda(\times_{i=1}^{d}) := \prod_{i-1}^{d} (b_i - a_i)$$

'hyper-volume', or volume for short. We call them all *Lebesgue measure*, and denote them all by λ (or λ_d if we need to display the dimension).

Note that by its definition λ is *translation-invariant*: if A is an interval/rectangle/cuboid etc., and $A + h := \{x + h : x \in A\}$, then

$$\lambda(A + h) = \lambda(A).$$

Such translation-invariant measures can be defined more generally in *topological groups*, giving *Haar measure* (Alfred HAAR (1885-1933), in 1933).

Completion.

Not all sets have a Lebesgue measure! – see below. But, if B has Lebesgue measure $\lambda(B) = 0$, and $A \subset B$, it would seem natural to say that $\lambda(A) = 0$ also – a subset of a set of length/area/volume 0 should also have length/area/volume 0.

Call a set A *μ-null* (or just *null*) if $\mu(A) = 0$. Call a measure space $(\Omega, \mathcal{A}, \mu)$ *complete* if the σ-field $\mathcal{A}$ of μ-measurable sets contains all subsets of null sets. It turns out that given a measure μ on a σ-field $\mathcal{A}$, one can always extend μ to a larger σ-field – the σ-field $\mathcal{A}^*$ generated by the sets in $\mathcal{A}$ and the null sets N. These are the sets of the form $A \Delta N$ with $A \in \mathcal{A}$ (i.e., A μ-measurable) and N null; the extension is given for such sets by

$$\mu(A \Delta N) := \mu(A).$$

For details, see e.g. [S], 29, 46.