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Measures (continued).

Completion (continued).

Usually (but not always) it is convenient and harmless to complete all measure spaces in this way. In this course, we shall assume completeness unless otherwise stated. We then have no need to distinguish between the σ-fields generated by the null sets N and by the subsets of null sets; we shall denote either by $\mathcal{N}$.

(Lebesgue)-measurable sets.

The class $\mathcal{L}$ of *Lebesgue-measurable sets* (*measurable sets* when the context is clear) is the completion of the class $\mathcal{B}$ of Borel sets (L2) by the class of null sets $\mathcal{N}$.

There are Lebesgue-measurable sets that are not Borel: $\mathcal{L} \setminus \mathcal{B}$ is non-empty. Indeed, 'most' Lebesgue-measurable sets are not Borel, in the sense of cardinality. We quote that if $\mathbf{c}$ is the cardinality of the continuum – the reals, or $[0, 1]$ – $\mathcal{B}$ has cardinality $\mathbf{c}$, but $\mathcal{L}$ has cardinality $2^{\mathbf{c}}$, which is much bigger.

Non-measurable sets.

Recall that a (non-empty) set A is *finite* if it cannot be put into one-one correspondence with a proper subset of itself. There is then a unique natural number N such that A can be put in one-one correspondence with $\{1, 2, \ldots, N\}$; N is called the *cardinality* of A, $N = |A|$ or $card(A)$. Otherwise A is *infinite*. Some infinite sets can be put in one-one correspondence with the natural numbers. These (the 'small' infinite sets) are called *countable* (or *denumerable* – though *listable* might be better, as one can list the elements of A as $A = \{a_1, a_2, \ldots, a_n, \ldots\}$). Examples: the integers, the rationals. The remaining infinite sets (most of them – the 'big' infinite sets) are not countable, and are called *uncountable* (examples: the real line; any interval of positive length).

Countability is built into Measure Theory in the property of countable additivity, (*ca*) (Lecture 3). But the real line is uncountable, so we must expect some problems with Measure Theory on the line, e.g. with Lebesgue measure.

Recall the *Axiom of Choice*, AC (Ernst ZERMELO (1871-1953), in 1904): from every collection of non-empty sets, it is possible to choose exactly one element from each set. This may seem obvious (it is not problematic if the collection is finite, nor even when the collection is countably infinite), but it is *not* possible to prove AC from ordinary Mathematics (technically, Zermelo-Fraenkel set theory, ZF – Zermelo in 1908, Abraham A. FRAENKEL (1891-1965) in the 1920s). We need it to do a number of branches of Analysis properly (e.g. Functional Analysis and Measure Theory), so when we need it, we assume it (usually in the form of Zorn's Lemma – Max ZORN (1906-1993), in 1935); ZF augmented by AC is written ZFC.

The problem with using AC, or Zorn's Lemma, is that proofs using them are *non-constructive*: they tell us something exists, but do not tell us how to find such things explicitly. Such non-constructive existence proofs are a fact of life in many areas of Mathematics, particularly Analysis.

Using AC, one can prove that non-(Lebesgue-)measurable subsets of the unit interval $[0, 1]$ exist. For $A \subset (0, 1]$ and $x \in (0, 1]$, write $E(x)$ for the set of points of $E + x$ (defined as above), reduced modulo 1 to lie in $(0, 1]$. By translation-invariance of Lebesgue measure, if A is Lebesgue-measurable, so is $A(x)$ and it has the same measure. Let

$$Z = \{r_1, r_2, \ldots, r_n, \ldots\}$$

be the (countable) set of rationals in $(0.1]$. One can check that two sets $Z(x_1)$ and $Z(x_2)$ are disjoint if $x_1 - x_2$ is irrational and identical if it is rational. Let $\mathcal{C}$ denote the class of all disjoint sets of the form $Z(x)$. By AC, there is a set T containing exactly one point from each of them. For each n, let

$$Q_n := T(r_n).$$

These are disjoint, and have union

$$\bigcup_{n=1}^{\infty} Q_n = (0, 1].$$

If T is Lebesgue-measurable, with measure $\lambda(T) = |T|$, then as above

$$|Q_n| = |T(r_n)| = |T|.$$

But if $|T| = c \geq 0$, countable additivity gives

$$1 = |(0, 1]| = |\bigcup_{n=1}^{\infty} Q_n| = \sum_{n=1}^{\infty} |Q_n| = \sum_{1}^{\infty} c.$$

On the left is 1, while on the right is a sum of infinitely many terms c, which is 0 if $c = 0$ and $+\infty$ if $c > 0$, a contradiction either way. So T cannot be Lebesgue-measurable. //

This example of a non-measurable set is due to Giuseppe VITALI (1875-1932) in 1905. It confirms what was suggested in L1 – that not all sets are measurable (have length/area/volume). Indeed, 'most' sets do not – a typical set is non-measurable.

But by above, we cannot construct ('get our hands on') such a typical set (we have to use AC, which is non-constructive). This is to be expected: the situation regarding typical sets (of reals) is similar to but more complicated than that regarding typical real numbers. 'Nice' reals are rational, or algebraic (e.g. $\sqrt{2}$, $\sqrt{3}$), or computable (e.g. π: one could write a computer programme to print out a sequence of approximations to it). But all these classes are countable, while there are uncountably many reals. So a typical real has none of these properties. Such non-constructive existence proofs are made possible by the work of Georg CANTOR (1845-1918) in 1874, and later on foundations of set theory.

'Almost everywhere' (a.e.).

The very name 'null set' for 'set of measure 0' suggests that what happens on a null set is exceptional, and less important than what happens off it. We say that a property holds *almost everywhere* (a.e.) if it happens except on some exceptional null set (French, p.p., presque partout, German f.u., fast überall).

Counting measure.

We note an example, much simpler than Lebesgue measure and in a sense diametrically opposite to it. Take Ω as the integers (or non-negative integers, or natural numbers/positive integers), with

$$\mu(A) := card(A)$$

(or $|A|$), the cardinality of a set A (number of points in it). This is trivially a measure, called *counting measure.* The formula above defines $\mu(A)$ for *all* subsets. So the σ-field for this measure space is $\mathcal{A} = \mathcal{P}(\Omega)$, the *power set* of Ω (class of *all* subsets of Ω). The measure space is *purely atomic* – has no non-trivial null sets (the only set of cardinality 0 is empty!). Counting measure is useful in Analysis, in the theory of infinite series, and in Probability and Statistics, when dealing with discrete distributions (binomial, Poisson etc.).

Dynkin systems and monotone classes.

A class $\mathcal{D}$ of subsets of Ω is called a *Dynkin system* (E. B. DYNKIN (1924 –)) if it is closed under complements and countable disjoint unions. For any class $\mathcal{G}$, there is a smallest Dynkin system containing it, written $\mathcal{D}(\mathcal{G})$ and called the Dynkin system *generated* by $\mathcal{G}$. Also

$$\mathcal{G} \subset \mathcal{D}(\mathcal{G}) \subset \sigma(\mathcal{G});$$

for proof and background, we refer to [S] Ch. 5. Further,
(i) a Dynkin system is a σ-field iff it is closed under (finite) intersections;
(ii) if $\mathcal{G}$ is closed under finite intersections, then $\mathcal{D}(\mathcal{G}) = \sigma(\mathcal{G})$.
Dynkin systems are important in that if two σ-finite measures agree on a Dynkin system, they agree on the generated σ-algebra. Since Dynkin systems are often easier to handle than σ-algebras, this is often convenient. For example, this is the basis of one way of proving uniqueness of Lebesgue measure.

Similarly, a class $\mathcal{M}$ is called a *monotone class* if it is closed under monotone (increasing or decreasing) limits of sets ($A_n \subset A_{n+1}$, or $A_n \supset A_{n+1}$). Again, given $\mathcal{G}$ there is a smallest monotone class $\mathcal{M}(\mathcal{G})$ containing it, and

$$\mathcal{G} \subset \mathcal{M}(\mathcal{G}) \subset \sigma(\mathcal{G}).$$

Again, if two σ-finite measures agree on a monotone class, they agree on the generated σ-algebra, and this is often convenient.

Inverse images

If we have a function f from a measure space $(\Omega_1, \mathcal{A}_1)$ to a measure space $(\Omega_2, \mathcal{A}_2)$, the *inverse image* f^{-1} maps sets $A \subset \Omega_2$ to

$$f^{-1}(A), \text{ or } f^{-1}A, := \{x \in \Omega_1 : f(x) \in A\}.$$

We note that inverse images respect set-theoretic operations:

$$f^{-1}(A^c) = (f^{-1}(A))^c, \quad f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n), \quad f^{-1}(\bigcap_n A_n) = \bigcap_n f^{-1}(A_n).$$

So writing $f^{-1}(\mathcal{A}) := \{f^{-1}(A) : A \in \mathcal{A}\}$, $f(\mathcal{A}) := \{f(A) : A \in \mathcal{A}\}$, $f^{-1}(\mathcal{A})$ is a σ-field if $\mathcal{A}$ is (and clearly $f(\mathcal{A})$ is a σ-field if $\mathcal{A}$ is).