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Lecture 5. 18.10.2010.

4. The Lebesgue integral.

4.1. The Riemann integral.

We begin with our first exposure to integration – the 'Sixth Form integral'. To find the area I under the graph of a bounded function $y = f(x)$ between $x = a$ and $x = b$, we *partition* the interval $[a, b]$ by a partition $\mathcal{P}$ of points x_i:

$$a = x_0 < x_1 < \ldots < x_n = b.$$

The area below the curve on $[x_i, x_{i+1}]$ is trapped between the areas of the 'rectangle below' and the 'rectangle above'. Summing, the required area is trapped between the sums of these, called the *lower Riemann sum* and the *upper Riemann sum*. As we refine the partition by adding more points, the lower sums increase (and are bounded above by any upper sum), so converge; similarly, the upper sums decrease, so converge. The limits are called the *lower Riemann integral* L and *upper Riemann integral* U; $L \leq U$. If $L = U$, we say that f is *Riemann integrable* (R-integrable) on $[a, b]$ with *Riemann integral* (R-integral)

$$\int_a^b f, \text{ or } \int_a^b f(x)dx, := L = U.$$

One can prove that, e.g,

(i) *continuous* functions are R-integrable;

(ii) *monotone* (increasing or decreasing) functions are R-integrable.

But (iii) Not all functions are R-integrable.

Example. $f(x) = I_{\mathbf{Q}}(x), := 1$ is x is rational, 0 if x is irrational, $a = 0$, $b = 1$. As each interval $[x_i, x_{i+1}]$ contains both rationals and irrationals, all lower R-sums are 0 and all upper R-sums are 1. So $L = 0$, $U = 1$, and f is not R-integrable.

But the situation is not symmetrical between rationals and irrationals. 'Most' reals are irrational (there are uncountably many irrationals but only countably many rationals). So $\int_0^1 I_{\mathbf{Q}}(x)dx$ 'ought' to be 1.

Question. Which functions are R-integrable?

We can answer this question, but it turns out that the answer involves Measure Theory. And if we have to learn Measure Theory, we might as well learn its counterpart, the integration theory that goes naturally with it. This is the *Lebesgue integral* (Lebesgue's thesis, 1902), below. This supercedes the R-integral: it is vastly more general, much easier to handle, and agrees with the R-integral whenever both are defined.

We quote:

Theorem. A bounded function f is Riemann integrable on $[a, b]$ iff f is continuous a.e. on $[a, b]$.

Note that this shows that the example above is as far away as possible from being R-integrable: $f = I_{\mathbf{Q}}$ is *discontinuous everywhere*.

It turns out that the essence of Lebesgue integration is to divide up the y-axis – the *values* of f – rather than the x-axis as in the Riemann integral above. This is neatly illustrated in the following anecdote about Lebesgue. Lebesgue's father was a shopkeeper, and Lebesgue prided himself on being down-to-earth and practical. At the end of each day, his father would total up the day's takings. There are two ways of doing this:

1. Keep a chronological record of each transaction. The day's total is the sum of the takings in each transaction (this would be done automatically in the machines in use at modern tills).

2. Go to the till. Count the number of each denomination of note (£50, £20, £10, £5) and each denomination of coins (£2, £1, 50p, 20p, 10p, 5p, 2p, 1p). The total is the sum of the totals for each denomination.

The second way is obviously far superior, both practically and conceptually. The second corresponds to the Lebesgue integral (below), the first to the Riemann integral (above).

4.2. Measurable functions.

If $f :\to (\Omega_2, \mathcal{A}_2)$ is a map between measurable spaces (i.e., if $f : \Omega_1 \to \Omega_2$) and Ω_i is endowed with a σ-field $\mathcal{A}_i$), call f *measurable* (mble) if

$$f^{-1}(A_2) := \{x : f(x) \in A_2\} \in \mathcal{A}_1$$

for all $A_2 \in \mathcal{A}_2$ – inverse images of measurable sets are measurable.

Unless otherwise stated, we specialize to real-valued functions and Borel sets:

$$f : (\Omega, \mathcal{A}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})).$$

It turns out that we do not have to test *all* Borel sets $B \in \mathcal{B}(\mathbf{R})$, just those in a family $\mathcal{G}$ that generates $\mathcal{B}(\mathbf{R})$. So f is mble iff

$$f^{-1}((-\infty, a]) := \{x : f(x) \leq a\} \in \mathcal{A}$$

for all real a, or even all rational a, and similarly for $<, \geq, >$ ([S], Lemma 8.1).

A mble function f is *simple* if it takes finitely many values (each on a mble set):

$$f = \sum_1^n c_i I_{A_i} \qquad (A_i \in \mathcal{A}).$$

Write $\mathcal{E}$, or $\mathcal{E}(\mathcal{A})$, for the class of simple functions ('e for elementary'). The representation $f = \sum c_i I_{A_i}$ is called *standard* if the A_i are disjoint. One can check that

(i) A measurable function is simple iff it takes only finitely many values.

(ii) If f, g are simple, so are $f \pm g$, fg and cf for c constant.

(iii) Write the positive and negative parts of f as

$$f^+ := \max(f, 0), \qquad f^- := -\min(f, 0)$$

– so

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-.$$

Then if f is simple, so is $|f|$.

Theorem. If f is measurable on $(\Omega, \mathcal{A})$, then f is a pointwise limit of simple functions f_n with $|f_n| \leq |f|$:

$$f_n(x) \to f(x) \qquad (n \to \infty).$$

If also $f \geq 0$, one can take $f_n \geq 0$ with

$$f_n(x) \uparrow f(x) \qquad (n \to \infty).$$

Proof (Sketch: see [S] Th. 8.8 for details). For $f \geq 0$, fix $n = 1, 2, \ldots$ and let

$$A_{k,n} := \{x : k/2^n \leq f(x) < (k+1)/2^n\} \quad (k < n.2^n), \quad \{x : f(x) \geq n\} \quad (k = n.2^n).$$

Define

$$f_n(x) := \sum_0^{n.2^n} k/2^n . I_{A_{k,n}}(x).$$

Then (check)

(i) $|f_n(x) - f(x)| \leq 2^{-n}$ if $f(x) < n$;

(ii) $A_{k,n} \in \mathcal{A}$;

(iii) $0 \leq f_n \leq f$ and $f_n \uparrow f$.

For general f, apply the case $f \geq 0$ to f^+ and f^- and use $f_n = f_n^+ - f_n^-$. //

If f_n is a sequence of measurable functions, then $\sup f_n$ is also measurable. For,

$$\{x : \sup_n f_n(x) > c\} = \bigcup \{x : f_n > c\}.$$

The sets on the RHS are mble (in $\mathcal{A}$) as f_n is mble; as $\mathcal{A}$ is a σ-field, so is their union; so the LHS is mble for each c, so $\sup_n f_n$ is mble. Similarly, $\inf_n f_n$ is mble. If f_n increases with n, then $\sup f_n = \lim f_n$, and similarly for decreasing sequences. So monotone limits of mble functions are mble. As

$$\text{limsup} f_n = \lim_{n \to \infty} \sup_{k \geq n} f_k,$$

$\limsup f_n$ is mble if the f_n, and similarly for $\liminf f_n$. In particular, if f_n are mble and $f_n \to f$, f is mble:

pointwise limits of measurable functions are measurable.

If $f : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ and $g : (\Omega_2, \mathcal{A}_2) \to (\Omega_3, \mathcal{A}_3)$ are measurable, the composite function $g \circ f = g(f(.)) : (\Omega_1, \mathcal{A}_1) \to (\Omega_3, \mathcal{A}_3)$ is measurable. For,

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A));$$

for $A \in \mathcal{A}_3$, $g^{-1}(A) \in \mathcal{A}_2$ as g is measurable, so $f^{-1}(g^{-1}(A)) \in \Omega_1$ as f is measurable:

compositions of measurable functions are measurable.

Image measures.

If $f : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ is measurable and μ is a measure on $(\Omega_1, \mathcal{A}_1)$, one can check that

$$\mu'(A) := \mu(f^{-1}(A)), \qquad A \in \mathcal{A}$$

defines a measure on $(\Omega_2, \mathcal{A}_2)$. It is called the *image measure* of μ under f, written $f(\mu)$ or $\mu \circ f^{-1}$:

$$f(\mu)(A) := \mu(f^{-1}(A)).$$