spl6.tex

Lecture 6. 22.10.2010.

4.3. The measure-theoretic integral; the Lebesgue integral.

If $f : (\Omega, \mathcal{A}, \mu) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is defined on a measure space, f is simple, $f = \sum c_i I_{A_i}$, we define

$$\int f d\mu := \sum c_i \mu(A_i).$$

Although the representation $f = \sum c_i I_{A_i}$ is not unique, $\sum c_i \mu(A_i)$ *is* unique. For details, see e.g. [S] Lemma 9.1, or Appendix 1.

When the measure μ is fixed, or understood, we can abbreviate $\int f d\mu$ to $\int f$. When μ is Lebesgue measure λ, we can write $\int f d\lambda$ as $\int f(x)dx$, or just $\int f$; we then call $\int f d\lambda$ the *Lebesgue integral* of f.

If $f \geq 0$, we can take f_n simple $\uparrow f$ (so each $\int f_n d\mu$ is defined), and write

$$\int f d\mu := \lim \int f_n d\mu$$

(the RHS is increasing in n as f_n is). If the limit $\int f d\mu$ is finite, we call f *μ-integrable*, with *μ-integral* $\int f d\mu$ (if $\int f d\mu = +\infty$, we say that f is not μ-integrable). By above, $\int f d\mu$ does not depend on the approximating simple sequence f_n.

For general f (not necessarily ≥ 0), $f = f^+ - f^-$; call f *μ-integrable* if f^+, f^- are μ-integrable; its *μ-integral* is then

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu.$$

Note that the RHS is well-defined, as both terms are finite: we must avoid meaningless expressions such as "$\infty - \infty$". Note that then

$$\int |f| d\mu := \int f^+ d\mu + \int f^- d\mu.$$

We summarise this by saying that the μ-integral is an *absolute integral*: f is μ-integrable iff $|f|$ is. We write $\mathcal{L}(\mu)$, or $\mathcal{L}_1(\mu)$, for the class of μ-integrable functions f ($\mathcal{L}$ for Lebesgue). For $p > 0$, write

$$\mathcal{L}_p(\mu) := \{f : f^p \in \mathcal{L}(\mu)\}$$

for the class of pth power integrable functions.

4.4. L_p-spaces.

Note that from the definition of the μ-integral, if $f = g$ μ-a.e., then $\int f d\mu = \int g d\mu$. For, altering f to g alters none of the terms $\mu(A_i)$ appearing in the approximations to $\int f d\mu$ by simple functions. In particular, if $f = 0$ μ-a.e., $\int f d\mu = 0$. If N is a μ-null set, applying this to $f.I_N$ gives

$$\int_N f d\mu = 0 \qquad \text{if } \mu(N) = 0.$$

Now when in mathematics we cannot tell two things apart, we should refuse to discriminate between them. Note that saying $f \equiv g$ iff $f = g$ μ-a.e. is an *equivalence relation*, $\sim$ (reflexive – $f \sim f$; symmetric – $f \sim g$ implies $g \sim f$; transitive – if $f \sim g$ and $g \sim h$, then $f \sim h$). Accordingly, we identify all functions equivalent to f with f – or, we pass from functions f to equivalence classes $[f]$ of functions. Since we shall always do this from now on, we write f instead of $[f]$ to simplify notation. But we change the notation for the classes $\mathcal{L}_p(\mu)$ of pth power integrable functions: when we pass to equivalence classes, we write these as $L_p(\mu)$, and call them the *L_p spaces*.

Note that in doing this we have abandoned an important part of the definition of a function. A function f with domain D and range R is a (single-valued) map from D to R. But now we no longer have individual values $f(x)$ for a function f at a point x.

For $p \geq 1$, the L_p-spaces have much structure and good properties. First, if $f \in L_p$, $cf \in L_p$ for c constant. Also, if $f, g \in L_p$, $f + g \in L_p$. This follows from *Minkowski's inequality*:

$$\left(\int |f + g|^p d\mu\right)^{1/p} \leq \left(\int |f|^p d\mu\right)^{1/p} + \left(\int |g|^p d\mu\right)^{1/p}$$

(Hermann MINKOWSKI (1864-1909) in 1896). So also $af + bg \in L_p$ for constants a, b: L_p is a *vector space*. For $f \in L_p$, write

$$\|f\|_p := \left(\int |f|^p d\mu\right)^{1/p}.$$

Then Minkowski's inequality becomes

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

This, together with $\|cf\|_p = |c|\|f\|_p$, is the defining property for a *norm*. Then $(L_p(\mu), \|.\|_p)$ becomes a *normed space*. It is a metric space, under the metric

$$d(f, g) := \|f - g\|_p,$$

hence in particular a topological space. The vector-space operations are continuous under this topology: we have a *topological vector space*. Define Cauchy sequences as in any metric space: (f_n) is Cauchy if

$$\|f_m - f_n\|_p = d(f_m, f_n) \to 0 \qquad (m, n \to \infty).$$

Recall that a metric space is *complete* if all Cauchy sequences are convergent (examples: the reals are complete, but the rationals are incomplete). Recall also that, just as we can construct the reals from the rationals via Cauchy sequences (Cantor's construction of the reals, 1872), we can similarly embed any metric space as a dense subset of a larger complete metric space, called its *completion*.

The following important result is the *Riesz-Fischer theorem* (F. RIESZ (1880-1956) and E. S. FISCHER (1875-1954), independently in 1907).

Theorem (Riesz-Fischer Theorem). The L_p spaces are complete.

A complete normed space is called a *Banach space* (after Stefan BANACH (1892-1945), *Théorie des opérations linéaires* in 1932). So the Riesz-Fischer theorem says that the *L_p-spaces are Banach spaces*. They are one prime class of examples of the *classical Banach spaces*, another being the spaces $C(K)$, the class of continuous functions on a compact topological space K.

5. Properties of the integral.

The following properties of the integral are elementary; see e.g. [S] for details. Here $f, g \in L(\mu)$, a, b are constant.

(i) If A, B are disjoint,

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

(ii) f is finite μ-a.e.

(iii) The integral is linear:

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

(iv) $|\int f d\mu| \leq \int |f| d\mu$.

(v) The integral is order-preserving:

$$f \geq 0 \Rightarrow \int f d\mu \geq 0; \qquad f \geq g \Rightarrow \int f d\mu \geq \int g d\mu.$$

In particular, if $|f| \le c$ on A, $|\int_A f d\mu| \le c\mu(A)$.
(vi) If $f \ge 0$ and $\int f d\mu = 0$, then $f = 0$ μ-a.e.
(vii) $f = g$ μ-a.e. implies $\int f d\mu = \int g dm$ (L4, repeated for emphasis).
(viii) If h is measurable and $|h| \le f$, then $h \in L(\mu)$ (and $|\int h d\mu| \le \int f d\mu$).

Theorem (Lebesgue's monotone convergence theorem, 1902: 'monotone convergence'). If f_n are non-negative measurable functions, $f_n \uparrow f$, then

$$\int f_n d\mu \uparrow \int f d\mu.$$

Here the conclusion means that if $f \in L(\mu)$, i.e. $\int f d\mu < \infty$, then both sides are finite; if not, the RHS is infinite, and the LHS $\uparrow \infty$ or is ∞ from some point on.

Proof. For each n, choose f_{nk} simple increasing to f_n as $k \to \infty$ (possible by L5). Then

$$g_k := \max_{n \le k} f_{nk}.$$

Then the g_k are increasing (with k), simple and non-negative, so

$$g \uparrow g \qquad (k \to \infty)$$

with g non-negative and measurable (as each g_k is). But for $n \le k$

$$f_{nk} \le g_k \le f_k \le f.$$

So letting $k \to \infty$,

$$f_n \le g \le f.$$

Letting $n \to \infty$, $f = g$. As the integral is order-preserving, by above

$$\int f_{nk} d\mu \le \int g_k d\mu \le \int f_k d\mu$$

for $n \le k$. Let $k \to \infty$: by definition of the integral (via simple approximations),

$$\int f_n d\mu \le \int g d\mu = \int f d\mu \le \lim_{k\to\infty} \int f_k d\mu$$

(as $g = f$). Let $n \to \infty$:

$$\lim_{n\to\infty} \int f_n d\mu \le \int f d\mu \le \lim_{k\to\infty} \int f_k d\mu.$$

As the two extremes are equal, these are equalities, which proves the result.
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