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Lecture 8. 25.10.2010.

Properties of the integral (continued).

The next result, Fatou's lemma, is due to Pierre FATOU (1878-1929) in 1906.

Theorem (Fatou's lemma). If f_n are measurable and bounded below by an integrable function g, then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. By passing to $f_n - g$, which is measurable and non-negative, and using linearity of the integral, it suffices to consider the case when $f_n \geq 0$. Put

$$g_n := \inf_{k\geq n} f_k;$$

then g_n is an increasing sequence, and

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n.$$

Now $f_n \geq g_n$. Take $\liminf \int$ and use that $\int g_n d\mu$ decreases (order property of the integral): the lim inf on the right is a lim, so

$$\liminf \int f_n d\mu \geq \lim \int g_n d\mu.$$

By monotone convergence, as g_n is increasing,

$$\lim \int g_n d\mu = \int \lim g_n d\mu = \int \liminf f_n d\mu.$$

Combining, the result follows. //

Of course, applying the result to $-f_n$ gives the alternative form: for functions bounded above by an integrable function, $\int \limsup \geq \limsup \int$.

Theorem (Lebesgue's dominated convergence theorem, 1910 – dominated convergence). If f_n are measurable, $f_n \to f$ and $|f_n| \leq g$ with g μ-integrable, then

$$\int f_n d\mu \to \int f d\mu.$$

Proof. First suppose $f_n \geq 0$ and $f_n \to 0$. By Fatou's lemma,

$$0 = \int 0 d\mu = \int \liminf f_n d\mu \leq \liminf \int f_n d\mu \leq \limsup \int f_n d\mu,$$

while by the 'limsup' form,

$$\limsup \int f_n d\mu \leq \int \limsup f_n d\mu = \int 0.d\mu = 0.$$

Combining these two ($0 = .. \leq ... \leq= 0$), each must be an equality. So $\lim \int f_n d\mu$ exists and is 0.

In the general case, as $|f_n| \leq g$ and $f_n \to f$, $|f| \leq g$; as g is μ-integrable, this gives f μ-integrable (L6, property (viii)). Put $g_n := |f_n - f|$. Then $0 \leq g_n \leq 2g$, $2g$ is μ-integrable, g_n is measurable and tends to 0. So

$$|\int f_n d\mu - \int f d\mu| \leq \int |f_n - f| d\mu \to 0,$$

by the first part applied to $g_n = |f_n - f|$. So $\int f_n d\mu \to \int f d\mu$. //

Note. 1. These convergence results on interchanging limit and integral – monotone and dominated convergence, and Fatou's lemma – are very powerful and useful, and form one of the main advantages of the Lebesgue (or more generally, measure-theoretic) integral. By contrast, such convergence results are known for the Riemann integral, but under much more stringent conditions. We quote: if f_n are R-integrable and $f_n \to f$ *uniformly* on $[a, b]$, then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

This condition of uniform convergence is too strong for this result to be much use in Real Analysis, where the Lebesgue integral is preferred because of the results above (L7).

By contrast, in Complex Analysis, the principal convergence result is that if f_n are holomorphic in a domain D in the complex plane, and $f_n \to f$ uniformly on compact subsets K of D, then
(i) f is holomorphic on D (this follows from Morera's theorem);
(ii) the derivatives converge: $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of D (this follows from the Cauchy integral formulae for derivatives).
The Riemann integral is adequate for most purposes in Complex Analysis.
2. We proved the three results in the usual [chronological] order – "M, F,

D". But they can be proved in the order "D, M, F", so are equivalent. See V. I. BOGACHEV, *Foundations of measure theory*, Vol. 1, 2, Springer, 2005.

6. Stieltjes integrals.

In both the Riemann integral and the Lebesgue integral, intervals $(a, b]$ played a major role, and we used that they have length $b - a$. The "dx" in $\int f(x)dx$ (Riemann or Lebesgue) comes from this. It turns out that we need to generalize this, and we can do so easily by the methods above.

Suppose that F is a non-decreasing function. Then F can have at worst jump discontinuities; we take F to be *right*-continuous at any jump points. We replace the length $b-a$ of $(a, b]$ by $F(b) - F(a)$. This gives a *set-function* μ_F, defined by

$$\mu_F((a, b]) := F(b) - F(a).$$

If in the Riemann integral we replace lower R-sums $\sum m_i(x_{i+1} - x_i)$ by $\sum m_i(F(x_{i+1}) - F(x_i))$, and similarly for upper R-sums, we obtain an extension of the R-integral, called the *Riemann-Stieltjes integral* or RS-integral (Thomas STIELTJES (1856- 1894) in 1894/5). It is written $\int_a^b f(x)dF(x)$; here f is called the *integrand*, F the *integrator*. Care is needed if both integrand and integrator can have common points of discontinuity. We shall need to allow F to have jumps; we restrict to f continuous accordingly.

If in the definition of the measure-theoretic integral we take μ to be μ_F on half-open intervals $(a, b]$ as above, and then construct the integral as with the Lebesgue integral but with $\mu_F((a, b]) := F(b) - F(a)$ in place of $b - a$, we obtain the *Lebesgue-Stieltjes integral* or LS-integral.

Such Stieltjes integrals are important in Probability Theory. As we shall see in Ch. II, a random variable (rv) (X say) has a (probability) distribution function, F $(= F_X)$. Then the LS-integral $\int g(x)dF(x)$ has the interpretation of an *expectation*, $Eg(X)$ of the function $g(X)$ of the rv X.

Signed measures.

We now drop the requirement that our set-functions be non-negative. While length/area/volume, probability and (gravitational) mass are all non-negative, electrostatic charge can have either sign. A *signed measure* is a countably additive set function (not necessarily non-negative). The measure theory of signed measures is fairly simple: a signed measure μ can be written uniquely as

$$\mu = \mu^+ - \mu^-,$$

where $\mu^\pm$ are measures, with disjoint supports (the support of a measure is the largest set whose complement is null). This is the *Hahn-Jordan theorem* Hans HAHN (1879-1934) in 1948, posth., Camille JORDAN (1838-1922) in 1881).

We can extend the LS integral from non-decreasing integrands F to

$$F = F_1 - F_2$$

that are the difference of two non-decreasing functions, in the obvious way:

$$\int f dF := \int f dF_1 - \int f dF_2$$

(both terms on the right must be finite – we must avoid '$\infty - \infty$'). This gives the LS integral with integrator the F, or the corresponding signed measure (cf. the Hahn-Jordan theorem).

So suitable integrators are differences of monotone functions. But how do we recognize them? For an interval $[a, b]$, let $\mathcal{P}$ be a partition:

$$a = x_0 < x_1 < \ldots < x_n = b.$$

For a function F, the *variation* of F over the partition $\mathcal{P}$ is

$$var(F, \mathcal{P}) := \sum |F(x_{i+1} - F(x_i)|.$$

Call F of *finite variation* (FV) on $[a, b]$ if

$$var_{[a,b]} F := \sup\{var(F, \mathcal{P})\} < \infty,$$

where $\mathcal{P}$ varies over all partitions. Of course, monotone functions are of FV: if F is monotone, $var(F, \mathcal{P}) = |F(b) - F(a)|$, so taking the sup over $\mathcal{P}$, $var_{[a,b]} F = |F(b) - F(a)|$. Of course also, we need to restrict to finite intervals (or compact sets): the case $F(x) \equiv x$ generating Lebesgue measure is the prototype, but x, though of FV on finite intervals, has infinite variation over the real line. We quote:

Theorem (Jordan, 1881). The following are equivalent:
(i) F is the difference of two monotone functions;
(ii) F is of finite variation (FV) on intervals $[a, b]$.

Later we will encounter *stochastic integrals* $\int h dX$, where both the integrand h and the integrator X are random (stochastic processes). These will be of two types: X of FV, when we can use LS-integrals as above, and X not FV (e.g.: Brownian motion, Ch. IV) when we will need an entirely new kind of integral, the *Itô integral*.