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STOCHASTIC PROCESSES: MOCK EXAMINATION 2010

Answer five questions out of six; 20 marks per question.

Q1. Show that Lebesgue measure is invariant under
(i) translations; [**4**]
(ii) rotations; [**4**]
(iii) the action of the Euclidean motion group. [**2**]

Show that the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $V = 4\pi abc/3$. [**10**]

Q2. State without proof Lebesgue's dominated convergence theorem. [**2**]

Prove its conditional form: that for $f_n \to f$ with $|f_n| \leq g \in L_1$, and $\mathcal{A}$ a σ-field,

$$E[f_n|\mathcal{A}] \to E[f|\mathcal{A}]. \qquad [\mathbf{8}]$$

Prove Scheffé's Lemma: that if f_n, f are probability densities with $f_n \to f$ a.e., and $\mathcal{B}$ is the Borel σ-field,

$$\sup_{B \in \mathcal{B}} |\int_B f_n - \int_B f| \leq \int |f_n - f| \to 0. \qquad [\mathbf{10}]$$

Q3. Define the *tail σ-field* of a stochastic process $X = (X_n)$. [**3**]

Prove *Kolmogorov's Zero-One Law*: that if the X_n) are independent, the probability of a tail event is 0 or 1. (You may quote that, from the Carathéodory extension procedure, if a measure μ is extended from a field $\mathcal{F}_0$ to the generated σ-field $\mathcal{F}$, μ may be approximated on $\mathcal{F}$ by its values on $\mathcal{F}_0$ in the sense that for any $A \in \mathcal{F}$ and $\epsilon > 0$ there is a set $A_0 \in \mathcal{F}_0$ such that $\mu(A \Delta A_0) < \epsilon$.) [**12**]

If A_n are independent events, and $A := \text{limsup} A_n$ is the event that infinitely many of the A_n occur, deduce that A has probability 0 or 1. [**3**]

State, without proof, when $P(A)$ is 0 and when it is 1. [**2**]

Q4. The bivariate normal distribution is defined by its density

$$f(x, y) = c \exp\{-\frac{1}{2} Q(x, y)\},$$

where $c = 1/(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2})$ is a constant (μ_i real, $\sigma_i > 0$, $-1 < \rho < 1$) and Q is the positive definite quadratic form

$$Q = \frac{1}{1-\rho^2}\Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{y-\mu_2}{\sigma_2}\Big) + \Big(\frac{y-\mu_2}{\sigma_2}\Big)^2\Big].$$

By completing the square in Q, or otherwise, show that
(i) the marginal distributions of X, Y are $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$; **[5]**
(ii) the conditional distribution of Y given $X = x$ is

$$N(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)). \qquad \textbf{[6]}$$

Verify the Conditional Mean Formula and the Conditional Variance Formula for this example, and interpret the parameter ρ. **[3, 3, 3]**

Q5. The *Lévy density* is

$$f(x) := \frac{1}{\sqrt{2\pi}}.\frac{1}{x^{3/2}}.\exp(-\frac{1}{2x}).$$

Show that it has Laplace transform $\phi(s) := \int_0^\infty e^{-sx}f(x)dx = e^{-\sqrt{2s}}$ (by finding $\phi'(s)$, and showing, by the change of variable

$$x = \frac{1}{2su}, \qquad \text{so} \qquad sx = \frac{1}{2u}, \qquad \frac{1}{2x} = su,$$

that ϕ satisfies the ODE $\phi'(s)/\phi(s) = -1/\sqrt{2s}$, or otherwise). **[10]**

If $X, X_1, \ldots, X_n$ are independent with this density, show that $(X_1 + \ldots + X_n)/n^2 =_d X$. Why does this not contradict the Strong Law of Large Numbers, or the Central Limit Theorem? **[3]**

Show how this density arises in connection with the stable subordinator (non-decreasing stable process) of index 1/2. **[7]**

Q6. Define a *convex function.* State without proof Jensen's inequality, and its conditional form. **[1, 2, 1]**
(i) If $M = (M_t)$ is a martingale, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submartingale. **[4]**
(ii) If in (i) M is a submartingale, and ϕ is also non-decreasing on the range of M, show that again $\phi(M)$ is a submartingale. **[4]**

(iii) Deduce that for $B = (B_t)$ Brownian motion, $B^2 = (B_t^2)$ is a submartingale. [**4**]
(iv) Find the increasing process in its Doob-Meyer decomposition. Deduce that Brownian motion has quadratic variation t. [**4**]

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