spexamsolns.tex

STOCHASTIC PROCESSES: MOCK EXAMINATION SOLUTIONS, 12.12.2010

Q1. Lebesgue measure λ is defined on the line by $\lambda((a,b]) := b - a$. This is translation-invariant: $\lambda((a+h, b+h]) = (b+h) - (a+h) = b - a = \lambda((a,b])$. Similarly in higher dimensions, λ is defined first on rectangles, cuboids etc.; the measure of each is the product of the measures of each side; each of these is translation-invariant by above; so Lebesgue measure in higher dimensions is translation-invariant, on the class of cuboids on which it is thus defined. This class forms a field; by Carathéodory's Extension Theorem Lebesgue measure extends uniquely to the generated σ-field, and translation-invariance goes over by approximation. It then extends to the σ-field of Lebesgue-measurable sets by completion, and again translation-invariance persists. Thus Lebesgue measure is translation-invariant, as required.

For rotation-invariance in the plane: Lebesgue measure could be defined on regions in polar coordinates of the form $r_1 < r \leq r_2$, $\theta_1 < \theta \leq \theta_2$ as $\pi[r_2^2 - r_1^2].(\theta_2 - \theta_1)/2\pi = \frac{1}{2}(r_2 + r_1)(r_2 - r_1)(\theta_2 - \theta_1)$ (or in differential form, by $dA = rdr.d\theta$), which is rotation-invariant as it depends only on the *difference* of the angle variables. Lebesgue measure so defined agrees with Lebesgue measure defined as above, by the uniqueness of the Carathéodory extension procedure in the σ-finite case, as here (Euclidean space is σ-finite). This extends to three dimensions using a similar argument on each of the Euler angles in spherical polar coordinates, and similarly to d dimensions.

Combining, Lebesgue measure is invariant under both translations and rotations. But these generate the group of Euclidean motions, so Lebesgue measure is also invariant under the action of this group.

For the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$: the volume V is the result of integrating the element of area $dV = dxdydz$ over the region $x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1$. The sphere of radius a has volume $V = 3\pi a^3/3$ (L1). One may reduce to this case by the linear transformation $(x, y, x) \mapsto (x, yb/a, zc/a)$. The image of the ellipsoid is now the sphere of radius a, with volume as above. The effect on the element of volume is $dV = dxdydz \mapsto dx.(dy.b/a).(dz.c/a) = dxdydz.bc/a^2$. This maps the volume V of the ellipsoid to $V_{sphere}.bc/a^2 = (4\pi a^3/3).bc/a^2 = 4\pi abc/3$. So $V = 4\pi abc/3$.

Q2 . Lebesgue's dominated convergence theorem states that for measurable functions $f_n \to f$ a.e. (w.r.t. a measure μ), with f_n dominated by a μ-integrable function g, then

$$\int f_n d\mu \to \int f d\mu.$$

To prove the conditional form of dominated convergence: choose $A \in \mathcal{A}$. By dominated convergence applied to $X_n I_A$,

$$\int_A X_n dP \to \int_A X dP.$$

By definition of conditional expectation, this says

$$\int_A E[X_n|\mathcal{A}]dP \to \int_A E[X|\mathcal{A}]dP.$$

As this holds for all $A \in \mathcal{A}$,

$$E[X_n|\mathcal{A}] \to E[X|\mathcal{A}]. \qquad //$$

To prove Scheffé's Lemma:

$$|\int_B f_n - \int_B f| = |\int_B (f_n - f)| \le \int_B |f_n - f|.$$

Taking sups over B proves the required inequality. Next, with $a \wedge b := \min(a, b)$,

$$|f_n - f| = f_n + f - 2 f_n \wedge f.$$

Integrate: $\int f_n = 1$, $\int f = 1$ as these are densities. As $0 \le f_n \wedge f \le f$, integrable, dominated convergence gives

$$\int f_n \wedge f \to \int f = 1.$$

So the integral of RHS $\to$ 1+1-2 = 0. So the integral of LHS $\to$ 0 also:

$$\int |f_n - f| \to 0. \quad //$$

Q3. The *tail σ-field* $\mathcal{T}$ of a process $X = (X_n)$ is the sub-σ-field of $\sigma(X)$ invariant under changes to finitely many of the X_n.

Write $\sigma_n(X) := \sigma(X_1, \ldots, X_n)$. Then $\sigma(X)$ is the σ-field generated by the increasing family $\sigma_n(X)$ of σ-fields. The union $\cup_n \sigma_n(X)$ forms a field, which generates the σ-field $\sigma(X)$. So (from the Carathéodory extension procedure, given), for $A \in \sigma(X)$ there are $A_n \in \sigma_n(X)$ with

$$P(A \Delta A_n) \to 0, \quad \textit{i.e.} \quad P(A \setminus A_n) \to 0, \quad P(A_n \setminus A) \to 0$$

(Δ is the symmetric difference). So (writing '$o(1)$' for 'term tending to 0')

$$P(A_n) = P(A_n \cap A) + P(A_n \setminus A) = P(A_n \cap A) + o(1),$$

and similarly

$$P(A) = P(A \cap A_n) + P(A \setminus A_n) = P(A_n \cap A) + o(1) = P(A_n) + o(1).$$

If $A \in \mathcal{T}$ is a tail event, A depends only on random variables X_k sufficiently far along (i.e. for k sufficiently large). As the X_n are independent, A is independent of each $\sigma_n(X)$. So

$$P(A \cap A_n) = P(A).P(A_n).$$

Let $n \to \infty$: by above, we get

$$P(A) = P(A).P(A) = P(A)^2.$$

So $x = P(A)$ satisfies the equation $x = x^2$, i.e. $x^2 - x = x(x-1) = 0$, whose roots are $x = 0$ or 1. So $P(A) = 0$ or 1: the probability of a tail event of a sequence of independent random variables is 0 or 1, proving Kolmogorov's Zero-One Law. //

If the events A_n are independent, their indicators I_{A_n} are independent random variables. The event

$$A := \text{limsup} A_n := \cap_n \cup_{m=n}^{\infty} A_m = \{A_n \ i.o.\}$$

that infinitely many A_n occur is a tail event. By Kolmogorov's Zero-One Law above, $P(A) = 0$ or 1. By the Borel-Cantelli Lemmas, $P(A) = 0$ or 1 according as $\sum P(A_n)$ converges or diverges.

Q4. We *complete the square*. We have the algebraic identity

$$(1-\rho^2)Q \equiv \Big[\Big(\frac{y-\mu_2}{\sigma_2}\Big) - \rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big]^2 + (1-\rho^2)\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2.$$

Then (taking the terms free of y out through the y-integral)

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(\frac{-\frac{1}{2}(y-c_x)^2}{\sigma_2^2(1-\rho^2)}\right) dy, \qquad (*)$$

where

$$c_x := \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1).$$

The integral is 1 ('normal density'). So $f_1(x) = \exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)/\sigma_1\sqrt{2\pi}$, which integrates to 1 ('normal density'). So $f(x,y)$ is a joint density function, with marginal density functions $f_1(x), f_2(y)$. So $f(x,y) = f_{X,Y}(x,y)$, $f_1(x) = f_X(x)$, $f_2(y) = f_Y(y)$.

Next, X, Y are normal: X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$. For, $f_1 = f_X$ is $N(\mu_1, \sigma_1^2)$ density above, and similarly for Y. This also gives $EX = \mu_1, EY = \mu_2, varX = \sigma_1^2, varY = \sigma_2^2$.

The conditional law of y given $X = x$ is $N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \quad \sigma_2^2(1-\rho^2))$. For, by completing the square (or, return to (*) with $\int$ and dy deleted):

$$f(x,y) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} . \frac{\exp(-\frac{1}{2}(y-c_x)^2/(\sigma_2^2(1-\rho^2)))}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}}.$$

The first factor is $f_1(x)$. So, $f_{Y|X}(y|x) = f(x,y)/f_1(x)$ is the second factor:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\Big(\frac{-(y-c_x)^2}{2\sigma_2^2(1-\rho^2)}\Big),$$

where c_x is the linear function of x given below (*).
So $var(Y|X) = \sigma_2^2(1-\rho^2)$, giving $E[var(Y|X)] = \sigma_2^2(1-\rho^2)$, and

$$E(Y|X=x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \quad var(Y|X=x) = \sigma_2^2(1-\rho^2):$$

$$E[E(Y|X)] = E[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X-\mu_1)] = \mu_2 = EY, \quad var[E(Y|X)] = (\rho\sigma_2/\sigma_1)^2.\sigma_1^2 = \rho^2\sigma_2^2.$$

Combining, this verifies the Conditional Mean Formula and the Conditional Variance Formula here, and shows that ρ^2 is the proportion of the variability of Y accounted for by knowledge of X.

Q5.
$$\phi(s) = \frac{1}{\sqrt{2\pi}}.\int_0^\infty \exp(-sx - \frac{1}{2x}).\frac{dx}{x^{3/2}}.$$
Differentiate under the integral sign (as we may, the integrand being monotone in s – we quote this):
$$\phi'(s) = -\frac{1}{\sqrt{2\pi}}.\int_0^\infty \exp(-sx - \frac{1}{2x}).\frac{dx}{\sqrt{x}}.$$
The change of variable suggested interchanges the two terms in the exponential. It reverses the limits, and (check)
$$\frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{2s}}.\frac{du}{u^{3/2}}.$$
This gives
$$\phi'(s) = -\frac{1}{\sqrt{2s}}.\phi(s): \qquad \frac{\phi'(s)}{\phi(s)} = -\frac{1}{\sqrt{2s}}.$$
Integrate: $\log\phi(s) = -\sqrt{2s} + c$, $\phi(s) = ce^{-\sqrt{2s}}$. But $\phi(0) = \int f = 1$, so $\phi(s) = e^{-\sqrt{2s}}$. //

Adding independent random variables multiplies Laplace transforms (as with CFs – from the Multiplication Theorem), so $X_1 + \ldots + X_n$ has Laplace transform $[\phi(s)]^n = e^{-n\sqrt{2s}}$. Replacing s by s/n^2, $(X_1 + \ldots + X_n)/n^2$ has Laplace transform $\phi(s) = e^{-\sqrt{2s}}$, the Laplace transform of X. So $(X_1 + \ldots + X_n)/n^2$ has the same distribution as X, as required.

This does not contradict the Strong Law of Large Numbers, as X has infinite mean. It does not contradict the Central Limit Theorem, as X has infinite variance.

The argument above shows that this law is infinitely divisible. So it corresponds to a Lévy process, $X = (X_t)$, and as the law of each X_t is concentrated on the positive half-line, the paths of X are increasing, that is, X is a subordinator. So we may use the Laplace rather than the Fourier form of the Laplace exponent: $E\exp\{-sX_t\} = \exp\{-t\sqrt{2s}\}$. Then
$$E\exp\{-sX(ct)\} = \exp\{-ct\sqrt{2s}\} = \exp\{-t\sqrt{2c^2s}\} = E\exp\{-tc^2X_t\}:$$
$X_{ct} =_d c^2X_t$. So $X_t =_d X_{ct}/c^2$: X is strictly stable with index $1/2$. So X is the stable subordinator with index $1/2$ (X is also the first-passage process of Brownian motion).

Q6. A function ϕ is *convex* if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \qquad \forall \lambda \in [0,1], x, y.$$

Jensen's inequality states that

$$\phi(E[X]) \leq E[\phi(X)]$$

for convex ϕ and random variables X with X, $\phi(X)$ both integrable. The *conditional Jensen inequality* states that for $\mathcal{A}$ a σ-field, ϕ, X as above,

$$\phi(E[X|\mathcal{A}]) \leq E[\phi(X)|\mathcal{A}].$$

(i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a martingale. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submartingale.
(ii) If M is a submartingale, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\begin{aligned} \phi(M_s) &\leq \phi(E[M_t|\mathcal{F}_s]), \\ &\leq E[\phi(M_t)|\mathcal{F}_s] \end{aligned}$$

by the conditional Jensen inequality again, and again $\phi(M)$ is a submartingale.
(iii) As Brownian motion B is a martingale (lectures), and x^2 is convex (its second derivative is $1 \geq 0$), B^2 is a submartingale by (i).
(iv) As $B_t^2 - t$ is a martingale (which you may quote here as it is not asked – but is easy to prove, as in lectures)

$$B_t^2 = [B_t^2 - t] + t \qquad \text{(submg = mg + incr)}$$

is the Doob-Meyer decomposition of B_t^2. The increasing process here is t, which is thus the quadratic variation of Brownian motion B.

N. H. Bingham