spl30.tex

Problems 1. 15.11.2010

Q1. Show that the *power set* $\mathcal{P}(\Omega)$ of all subsets of a set Ω forms a ring (the *Boolean ring*) with addition as symmetric difference Δ and multiplication as intersection $\cap$. Find the zero element 0 and the identity element 1. Find also the additive inverse of each set $A \subset \mathcal{P}(\Omega)$.

Q2. With the convention that the sup of a set of reals unbounded above is $+\infty$ and the inf of a set unbounded below is $-\infty$,

$$\limsup_{n\to\infty} x_n := \inf_n \sup_{k\geq n} x_k, \qquad \liminf_{n\to\infty} x_n := \sup_n \inf_{k\geq n} x_k$$

are always defined (possibly $\pm\infty$).
(i) If $A_n \subset \Omega$,

$$\limsup A_n := \cap_n \cup_{k\geq n} A_k, \qquad \liminf A_n := \cup_n \cap_{k\geq n} A_k.$$

Show that

$$\limsup A_n = \{x : x \in A_n \text{ for infinitely many } n\}$$

(or $\{A_n$ infinitely often$\}$, or $\{A_n$ *i.o.*$\}$,

$$\liminf A_n = \{x : x \in A_n \text{ for all sufficiently large } n\}.$$

(ii) Show that

$$I_{\limsup A_n} = \limsup I_{A_n}, \qquad I_{\liminf A_n} = \liminf I_{A_n}.$$

Q3. Show that the union of countably many countable sets is countable.

Q4. Show that the union of countably many μ-null sets is μ-null.

NHB