spl30.tex

Problems 3. 29.11.2010

Q1. A function f is *convex* if for $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

for all x_i. We quote:
(a) this is equivalent to

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n \lambda_2 f(x_n)$$

for all n and all x_i (*Jensen's Inequality*; $\lambda_1 x_1 + \ldots + \lambda_n x_n$ is called a *convex combination* of the x_i);
(b) for twice differentiable functions f, $f'' \geq 0$ implies convexity.
(i) Draw a picture, and interpret convexity as 'chord below arc' (concavity is 'arc below chord').
(ii) Show that $-\log\ x$ is convex on $(0, \infty)$. Hence show that for $\lambda_i \geq$ and summing to 1, $x_i > 0$,

$$x_1^{\lambda_1} x_2^{\lambda_2} \leq \lambda_1 x_1 + \lambda_2 x_2.$$

(iii) Show that e^x is convex. Deduce that for $x_i > 0$ and λ_i as above,

$$x_1^{\lambda_1}. \ldots .x_n^{\lambda_n} \leq \lambda_1 x_1 + \ldots + \lambda_n x_n.$$

Take each $\lambda_i = 1/n$ to obtain

$$(x_1. \ldots .x_n)^{1/n} \leq (x_1 + \ldots + x_n)/n.$$

(The LSE is called the *geometric mean*, G; the RHS is called the *arithmetic mean*, A; this result $G \leq A$ is called the *AM-GM inequality*.)

Q2. *Hölder's inequality.* If $f \in L_p$, $g \in L_q$, $p > 1$ (or $q > 1$), and p, q are conjugate indices,

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then (i) $fg \in L_1$;
(ii)

$$\int |fg| \leq (\int |f|^p)^{1/p}.(\int |g|^q)^{1/q} : \qquad \|fg\|_1 \leq \|f\|_p.\|g\|_q.$$

When $p = q = 2$, deduce the *Cauchy-Schwarz inequality*: if $f, g \in L_2$, then $fg \in L_1$ and

$$\int |fg| \leq \sqrt{(\int |f|^2).(\int |g|^2)} : \qquad \|fg\|_1 \leq \|f\|_2.\|g\|_2.$$

Q3. *Minkowski's inequality*. If $p \geq 1$, and $f, g \in L_p$, then
(i) $f + g \in L_p$, and
(ii)

$$(\int |f+g|^p d\mu)^{1/p} \leq (\int |f|^p d\mu)^{1/p} + (\int |g|^p d\mu)^{1/p} : \quad \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

NHB