spprob4.tex

Problems 4. 5.11.2010

Q1. Means, variances etc. under affine transformations. Show that:

- (i) Means: E[aX + b] = aEX + b;
- (ii) Variances: $var[aX + b] = a^2 var[X];$
- (iii) SDs: SD[aX + b] = |a|SD[X];
- (iv) Covariances: $cov[aX + b, aY + b] = a^2 cov[X, Y];$
- (v) Correlations: corr[aX + b, aY + b] = corr[X, Y].

State and prove the corresponding results for sample means etc.

Q2. Central moments. Write $\mu_k := E[X^k]$, $\mu_{k,0} := E[(X - EX)^k]$ for the kth moment and central moment of a random variable X. Define the sample central moments, and show that they converge to the (population) central moments (as the sample size n increases).

Q3. Cumulants. The moment-generating function (MGF) and cumulantgenerating function (CGF) are

$$M(t) := E[e^{tX}] = \sum_{0}^{\infty} \mu_k t^k / k!, \qquad K(t) := \log M(t) = \sum_{1}^{\infty} \kappa_k t^k / k! = \kappa_1 t + \dots$$

Their centred versions have X replaced by X - EX, μ_k , κ_k by $\mu_{k,0}$, $\kappa_{k,0}$. Show that (i) $\kappa_1 = \mu$, $\kappa_k = \kappa_{k,0}$ for k = 2, 3, ..., (ii) $\kappa_2 = \sigma^2$, (iii) $\kappa_3 = \mu_{3,0}$, (iv) $\kappa_{4,0} = \mu_{4,0} - 3\sigma^4$. Show also that X is normal iff all cumulants above the second vanish.

Q4. Skewness and kurtosis. Define the skewness γ_1 and the kurtosis γ_2 by

$$\gamma_1 := \kappa_3 / \kappa_2^{3/2} = \mu_{3,0} / \sigma^3, \qquad \gamma_2 := \kappa_4 / \kappa_2^2 = \frac{\mu_{4,0}}{\sigma^4} - 3$$

Show that (i) $\gamma_1[aX+b] = \gamma_1[X], \gamma_2[aX+b] = \gamma_2[X]$; (ii) the sample central moments satisfy $\mu_{k,0} \to \mu_{k,0}$ as the sample size *n* increases; (iii) the sample skewness and kurtosis converge: $\hat{\gamma}_1 \to \gamma_1, \hat{\gamma}_2 \to \gamma_2$.

Note. Skewness measures asymmetry, kurtosis measures thickness of tails, both crucially important for financial data; we can estimate them by (iii).

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